Perturbed Iterate Analysis for Asynchronous Stochastic Optimization

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Abstract

We introduce and analyze stochastic optimization methods where the input to each gradient update is perturbed by bounded noise. We show that this framework forms the basis of a unified approach to analyze asynchronous implementations of stochastic optimization algorithms. In this framework, asynchronous stochastic optimization algorithms can be thought of as serial methods operating on noisy inputs. Using our perturbed iterate framework, we provide new analyses of the HOGWILD! algorithm and asynchronous stochastic coordinate descent, that are simpler than earlier analyses, remove many assumptions of previous models, and in some cases yield improved upper bounds on the convergence rates. We proceed to apply our framework to develop and analyze KROMAGNON: a novel, parallel, sparse stochastic variance-reduced gradient (SVRG) algorithm. We demonstrate experimentally on a 16-core machine that the sparse and parallel version of SVRG is in some cases more than four orders of magnitude faster than the standard SVRG algorithm.

1 Introduction

Asynchronous parallel stochastic optimization algorithms have recently gained significant traction in algorithmic machine learning. A large body of recent work has demonstrated that near-linear speedups are achievable, in theory and practice, on many common machine learning tasks [1–8]. Moreover, when these lock-free algorithms are applied to non-convex optimization, significant speedups are still achieved with no loss of statistical accuracy. This behavior has been demonstrated in practice in state-of-the-art deep learning systems such as Google's Downpour SGD [9] and Microsoft's Project Adam [10].

Although asynchronous stochastic algorithms are simple to implement and enjoy excellent performance in practice, they are challenging to analyze theoretically. The current analyses require lengthy derivations and several assumptions that sometimes may not reflect realistic system behaviors. Moreover, due to the difficult proof machinery, the algorithms analyzed are often simplified versions of the ones actually run in practice. To overcome these difficulties, we propose a general framework for obtaining convergence rates for parallel, lock-free, asynchronous first-order stochastic algorithms. We interpret the algorithmic effects of asynchrony as perturbing the stochastic iterates with bounded noise. This intuition allows us to show how a variety of asynchronous first-order algorithms can be analyzed as their serial counterparts operating on noisy inputs. The advantage of our framework is that it is compact and elementary, can remove or relax simplifying assumptions adopted in prior art, and can yield improved bounds than earlier work.

We demonstrate the general applicability of our framework by providing new convergence analyses for HOGWILD!, *i.e.*, the asynchronous stochastic gradient method (SGM), for asynchronous stochastic coordinate descent (ASCD), and KROMAGNON: a new asynchronous sparse version of the stochastic variance-reduced gradient (SVRG) method [11]. In particular, we provide a modified version of SVRG that allows for sparse updates, we show that this method can be parallelized in the asynchronous model, and then we provide convergence guarantees using our framework. Experimentally, the asynchronous, parallel sparse SVRG achieves nearly-linear speedups on a machine with 16 cores and is sometimes four orders of magnitude faster than the standard (dense) SVRG.

1.1 Related work

The algorithmic tapestry of parallel stochastic optimization is rich and diverse extending back at least to the late 60s [12]. Much of the contemporary work in this space is built upon the foundational work of Bertsekas, Tsitsiklis et al. [13, 14]. Recent advances in parallel and distributed computing technologies have generated renewed interest in the theoretical understanding and practical implementation of parallel stochastic algorithms [15–20].

The power of lock-free and asynchronous stochastic optimization on shared-memory multicore systems was first demonstrated in the work of [1]. The authors introduce HOGWILD!, a completely lock-free and asynchronous parallel stochastic gradient method (SGM) that exhibits nearly linear speedups for a variety of machine learning tasks. Inspired by HOGWILD!, several authors developed lock-free and asynchronous algorithms that move beyond SGM, such as the work of Liu et al. on parallel stochastic coordinate descent [5, 21]. Additional work in first order optimization and beyond [6–8, 22, 23], extending to parallel iterative linear solvers [24, 25], has further demonstrated that linear speedups are generically possible in the asynchronous shared memory model.

While this paper was in preparation, several new independent studies on the analysis of asynchronous optimization algorithms have appeared [26–29]: [26] proposes an analysis for asynchronous parallel, but dense, SVRG, under assumptions similar to those found in [1]. The authors of [27] offer a new analysis for the "coordinate-wise" update version of HOGWILD! using martingales, with similar assumptions to [1], that however does not require any sparsity assumptions on the gradient updates, and can be applied to some non-convex problems. Furthermore, [28] presents an analysis for stochastic gradient methods on smooth, potentially nonconvex functions. Finally, [29] introduces a new framework for analyzing coordinate-wise fixed point stochastic iterations.

2 Perturbed Stochastic Gradients

Preliminaries and Notation We study parallel asynchronous iterative algorithms that minimize convex functions $f(\mathbf{x})$ with $\mathbf{x} \in \mathbb{R}^d$. The computational model is the same as that of Niu et al. [1]: a number of cores have access to the same shared memory, and each of them can read and update components of $\mathbf{x}$ in the shared memory. The algorithms that we consider are asynchronous and lock-free: cores do not coordinate their reads or writes, and while a core is reading/writing other cores can update the shared variables in $\mathbf{x}$.

We focus our analysis on functions f that are L-smooth and m-strongly convex. A function f is L-smooth if it is differentiable and has Lipschitz gradients

$$\|\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\| \leq L\|\mathbf{x} - \mathbf{y}\| \text{ for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^d \tag{2.1}$$

where $\|\cdot\|$ denotes the Euclidean norm. Strong convexity with parameter $m > 0$ imposes a curvature condition on f:

$$f(\mathbf{x}) \geq f(\mathbf{y}) + \langle \nabla f(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle + \frac{m}{2}\|\mathbf{x} - \mathbf{y}\|^2 \text{ for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^d. \tag{2.2}$$

Strong convexity implies that f has a unique minimum $\mathbf{x}^*$, and that the following inequality holds

$$\langle \nabla f(\mathbf{x}) - \nabla f(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle \geq m\|\mathbf{x} - \mathbf{y}\|^2. \tag{2.3}$$

In the following, we use i, j, and k to denote iteration counters, while reserving v and u to denote coordinate indices. We use $\mathcal{O}(1)$ to denote absolute constants.

Perturbed Iterates A popular way to minimize a convex functions is via *first-order stochastic* algorithms. These algorithms can be described using the following iteration

$$\mathbf{x}_{j+1} = \mathbf{x}_j - \gamma \mathbf{g}(\mathbf{x}_j, \xi_j) \tag{2.4}$$

where ξ_j is a random variable independent of $\mathbf{x}_j$ and $\mathbb{E}_{\xi_j} \mathbf{g}(\mathbf{x}_j, \xi_j) = \nabla f(\mathbf{x}_j)$ is an unbiased estimator of the true gradient of f at $\mathbf{x}_j$. The success of first-order stochastic techniques partly lies in their computational efficiency: the cost of using noisy estimates of the gradient is dwarfed by that of exactly computing it.

A major advantage of the above iterative formula is that—in combination with strong convexity, and smoothness inequalities—one can easily track the algorithmic progress and establish convergence rates to the optimal solution. Unfortunately, the progress of asynchronous parallel algorithms cannot be precisely described, or analyzed using the above iterative framework. Processors do not read from memory actual iterates $\mathbf{x}_j$, as there is no global clock that synchronizes reads or writes while different cores write/read "stale" variables.

In the subsequent sections, we show that the following simple perturbed variant of Eq. (2.4) can capture the algorithmic progress of asynchronous stochastic algorithms. Consider the following iteration

$$\mathbf{x}_{j+1} = \mathbf{x}_j - \gamma \mathbf{g}(\mathbf{x}_j + \mathbf{n}_j, \xi_j) \tag{2.5}$$

where $\mathbf{n}_j$ is a stochastic error term. For simplicity let $\hat{\mathbf{x}}_j = \mathbf{x}_j + \mathbf{n}_j$.[1] Then,

$$\begin{aligned} \|\mathbf{x}_{j+1} - \mathbf{x}^*\|^2 &= \|\mathbf{x}_j - \gamma \mathbf{g}(\hat{\mathbf{x}}_j, \xi_j) - \mathbf{x}^*\|^2 \\ &= \|\mathbf{x}_j - \mathbf{x}^*\|^2 - 2\gamma \left\langle \mathbf{x}_j - \mathbf{x}^*, \mathbf{g}(\hat{\mathbf{x}}_j, \xi_j) \right\rangle + \gamma^2 \|\mathbf{g}(\hat{\mathbf{x}}_j, \xi_j)\|^2 \\ &= \|\mathbf{x}_j - \mathbf{x}^*\|^2 - 2\gamma \left\langle \hat{\mathbf{x}}_j - \mathbf{x}^*, \mathbf{g}(\hat{\mathbf{x}}_j, \xi_j) \right\rangle + \gamma^2 \|\mathbf{g}(\hat{\mathbf{x}}_j, \xi_j)\|^2 + 2\gamma \left\langle \hat{\mathbf{x}}_j - \mathbf{x}_j, \mathbf{g}(\hat{\mathbf{x}}_j, \xi_j) \right\rangle \end{aligned} \tag{2.6}$$

where in the last equation we added and subtracted the term $2\gamma \left\langle \hat{\mathbf{x}}_j, \mathbf{g}(\hat{\mathbf{x}}_j, \xi_j) \right\rangle$. We assume that $\hat{\mathbf{x}}_j$ and ξ_j are independent.[2] However, in contrast to recursion (2.4), we do not require $\mathbf{x}_j$ to be independent of ξ_j anymore.

We now take expectation of both sides of (2.6). Since $\hat{\mathbf{x}}_j$ and $\mathbf{x}^*$ are independent of ξ_j, we use the property of iterated expectations to obtain

$$\mathbb{E} \left\langle \hat{\mathbf{x}}_j - \mathbf{x}^*, \mathbf{g}(\hat{\mathbf{x}}_j, \xi_j) \right\rangle = \mathbb{E} \left\langle \hat{\mathbf{x}}_j - \mathbf{x}^*, \nabla f(\hat{\mathbf{x}}_j) \right\rangle .$$

Moreover, since f is m-strongly convex, we know that

$$\langle \hat{\mathbf{x}}_j - \mathbf{x}^*, \nabla f(\hat{\mathbf{x}}_j) \rangle \geq m \|\hat{\mathbf{x}}_j - \mathbf{x}^*\|^2 \geq \frac{m}{2} \|\mathbf{x}_j - \mathbf{x}^*\|^2 - m \|\hat{\mathbf{x}}_j - \mathbf{x}_j\|^2, \tag{2.7}$$

where the second inequality is a simple consequence of the triangle inequality. Let $a_j = \mathbb{E}\|\mathbf{x}_j - \mathbf{x}^*\|^2$ and substitute (2.7) back into Eq. (2.6) to get

$$a_{j+1} \leq (1 - \gamma m) a_j + \gamma^2 \underbrace{\mathbb{E}\|g(\hat{\mathbf{x}}_j, \xi_j)\|^2}_{R_0^j} + 2\gamma m \underbrace{\mathbb{E}\|\hat{\mathbf{x}}_j - \mathbf{x}_j\|^2}_{R_1^j} + 2\gamma \underbrace{\mathbb{E} \left\langle \hat{\mathbf{x}}_j - \mathbf{x}_j, \mathbf{g}(\hat{\mathbf{x}}_j, \xi_j) \right\rangle}_{R_2^j} . \tag{2.8}$$

The recursive equation (2.8) is key to our analysis. We show that for given R_0^j, R_1^j, R_2^j, we can precisely establish convergence rates through elementary algebraic manipulations. Observe that there are three "error" terms in (2.8): R_0^j captures the stochastic gradient decay with each iteration, R_1^j captures the mismatch between the true iterate and the noisy estimate, and R_2^j is the projection of that mismatch on the gradient at each step. The key contribution of our work is to show that 1) this iteration can capture the algorithmic progress of asynchronous algorithms, and 2) the error terms can be bounded to obtain a $\mathcal{O}(\log(1/\epsilon)/\epsilon)$ rate for HOGWILD!, and linear rates of convergence for asynchronous SCD, and asynchronous sparse SVRG.

3 Analyzing Hogwild!

We provide a simple analysis of HOGWILD!. In this section we assume that f is decomposable in n parts

$$f(\mathbf{x}) = \frac{1}{n} \sum_{i=1}^{n} f_{e_i}(\mathbf{x}) \tag{3.1}$$

where $\mathbf{x} \in \mathbb{R}^d$, and each $f_{e_i}(\mathbf{x})$ depends only on the coordinates indexed by the subset e_i of $\{1, 2, \ldots, d\}$[3].

We refer to the sets e_i as *hyperedges* and denote the set of hyperedges by $\mathcal{E}$. We sometimes refer to f_{e_i}s as f's terms. The hyperedges imply a bipartite graph between the n terms and the d variables in $\mathbf{x}$, and a conflict graph between the n function terms, as shown in Fig. 1.

[1] The notion of gradient perturbation has been used in different contexts in the past (e.g., see [30] and references therein).

[2] This assumption might seem technical, however it will become clear in the next section why we need it.

[3] for simplicity we assume that f's terms are differentiable; our results can be extended to non-differentiable f_{e_i}.

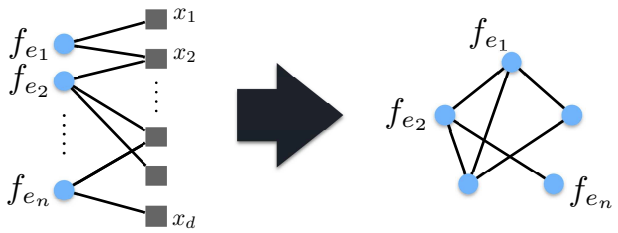


Figure 1: The left bipartite graph has as left vertices the n function terms, and as right vertices the coordinates of $\mathbf{x}$. A term f_{e_i} is connected to a coordinate x_j, if hyperedge e_i contains j (*i.e.*, if the i-th term is a function of that coordinate). The left graph denotes a conflict graph between the function terms. The vertices denote the function terms, and two terms are joined by an edge if they conflict on at least one coordinate in the bipartite graph.

As we will see, under our perturbed iterate analysis framework the convergence rate of asynchronous algorithms depends on the sparsity of the problem. Let us define by $\overline{\Delta}_{\mathrm{C}}$, the average degree in the conflict graph, which denotes the average number of terms that are in conflict with a single term. We assume that $\overline{\Delta}_{\mathrm{C}} \geq 1$, otherwise we could decompose the problem into smaller independent sub-problems.

HOGWILD! (Alg. 1) is a method to parallelize SGM [1]. It is deployed on multiple cores that have access to shared memory, where the optimization variable $\mathbf{x}$ and the data points that define the f terms are stored. During its execution each core samples uniformly at random a hyperedge s from $\mathcal{E}$. It reads the coordinates $v \in s$ of the shared vector $\mathbf{x}$, evaluates ∇f_s at the point read, and finally adds $-\gamma \nabla f_s$ to the shared variable.

Algorithm 1 HOGWILD!

1: **while** number of sampled edges $\leq T$ **do in parallel**
2: sample a random hyperedge s
3: $[\hat{\mathbf{x}}]_s$ = an inconsistent read of the shared variable $[\mathbf{x}]_s$
4: $[\mathbf{u}]_s = -\gamma \cdot \mathbf{g}([\hat{\mathbf{x}}]_s, s)$
5: **for** $v \in s$ **do**
6: $[\mathbf{x}]_v = [\mathbf{x}]_v + [\mathbf{u}]_v$ // atomic write
7: **end for**
8: **end while**

In HOGWILD! cores do not synchronize or follow an order between reads or writes. Moreover, they access/update a set of coordinates in $\mathbf{x}$ without the use of any locking mechanisms that would ensure a conflict-free execution. The reads/writes of distinct cores can intertwine in arbitrary ways, *e.g.*, while a core updates a subset of variables, before completing its task, other cores can access/update the same variables.

In [1], the authors analyzed a simplified variant of HOGWILD! where only a single coordinate per sampled hyperedge is updated (*i.e.*, the for loop in HOGWILD! is replaced with a single coordinate update). This simplification along with others, such as the assumption of consistent reads, alleviate some of the challenges in analyzing the algorithm and allowed the authors to provide a convergence analysis. We show how our perturbed gradient framework can be used in an elementary way to analyze the "full updates" version of HOGWILD!, while obtaining an improved bound compared to [1].

In the following, we denote by s_i the i-th sampled hyperedge by Alg. 1 that is assigned to some core, and by $\hat{\mathbf{x}}_i \in \mathbb{R}^d$ the contents of the shared memory read by this core; hence, the read variable vector "inherits" the same iteration index as the sampled hyperedge s_i. Since the core reads only the coordinates indexed by s_i, we need to clarify what is $[\hat{\mathbf{x}}_i]_v$ for coordinates that are not in the hyperedge s_i, *i.e.* for $v \notin s_i$. All the bounds presented in this work hold even if $[\hat{\mathbf{x}}_i]_v$ for every $v \notin s_i$, is equal to *any possible* values stored in the shared memory at position v during the processing of s_i. For the purpose of analysis, the reader can think of a processor reading all coordinates $v \notin s_i$, after sampling s_i. We note that we do not assume *consistent reads*, *i.e.*, the shared variables in memory can potentially change while a core is reading them.

Assumption 1. *The iterate $\hat{\mathbf{x}}_i$ is independent of the sampled hyperedge s_i.*

This assumption may appear trivial and it is indeed made implicitly or explicitly in much of the previous work (*e.g.*, [1, 5, 6]). However, it does not hold when the cores first sample the hyperedge and then read the coordinates of $\mathbf{x}$ indexed by that hyperedge. For example, consider the scenario of two consecutively sampled hyperedges in Alg. 1 that overlap on a specific set of coordinates. Then, say one core is reading the coordinates of the shared variables indexed by its hyperedge, while the second core is updating a subset of these coordinates. In this case, the contents of the variables read by the first core depend on the support of the sampled hyperedge. We can rigorously enforce Assumption 1 by requiring each core to read the *entire* shared memory vector before sampling a hyperedge (*i.e.*, by switching the order of lines 2 and 3 in Alg. 1). However, this is an unreasonable thing to do in practice, as the dimension of $\mathbf{x}$ tends to be considerably larger than the sparsity of the hyperedges.

We use an additional key assumption that is common in this literature.

Assumption 2 (Atomic writes)**.** *Every write in line 6 of Alg. 1 will complete successfully.*

Atomic addition, combined with its commutative nature, implies that all writes will appear in the shared memory by the end of the execution, in the form of additive coordinate-wise updates; due to commutativity the order of these updates is irrelevant. Hence, the following vector is contained in the shared memory after processing a total of T hyperedges:[4]

$$\underbrace{\overbrace{\mathbf{x}_0 - \gamma \mathbf{g}(\hat{\mathbf{x}}_0, s_0)}^{\mathbf{x}_1} - \ldots - \gamma \mathbf{g}(\hat{\mathbf{x}}_{T-1}, s_{T-1})}_{\mathbf{x}_T}, \tag{3.2}$$

where $\mathbf{x}_0$ is the initial guess. We now define the perturbed iterates as

$$\mathbf{x}_{i+1} = \mathbf{x}_i - \gamma \mathbf{g}(\hat{\mathbf{x}}_i, s_i) \tag{3.3}$$

for $i = 0, 1, ..., T-1$, where s_i is the i-th uniformly sampled hyperedge. Observe that all but the last of these iterates are "fake": there might not be an actual time when they exist in the shared memory during the execution. However, $\mathbf{x}_0$ is what is stored in memory before the execution starts, and $\mathbf{x}_T$ is exactly what is stored in shared memory at the end of the execution.

We observe that the above iterates place HOGWILD! in our perturbed gradient framework introduced in Section 2:

$$a_{j+1} \leq (1 - \gamma m) a_j + \gamma^2 \underbrace{\mathbb{E}\|g(\hat{\mathbf{x}}_j, s_j)\|^2}_{R_0^j} + 2\gamma m \underbrace{\mathbb{E}\|\hat{\mathbf{x}}_j - \mathbf{x}_j\|^2}_{R_1^j} + 2\gamma \underbrace{\mathbb{E}\left\langle \hat{\mathbf{x}}_j - \mathbf{x}_j, \mathbf{g}(\hat{\mathbf{x}}_j, s_j) \right\rangle}_{R_2^j}.$$

We are only left to bound the three error terms R_0^j, R_1^j, and R_2^j. Before we proceed, we present a simple definition that is required for technical soundness of our theorems, and an assumption on the delays between the cores.

Definition 1 (System Randomness)**.** *We define ξ, a random variable that encodes the randomness of the system (i.e., random delays between reads and writes, gradient computation, etc). The random variables $s_0, \ldots, s_{T-1}, \xi$ completely determine the time of all reads and writes.*

We note that we never use explicitly the random variable ξ. However, it is required as the random samples s_0, s_1, ..., s_{T-1} do not fully determine the output of Alg. 1, and is further required to rigorously assume a property on the delay of processing among cores.

Assumption 3 (bounded overlaps)**.** *Two hyperedges s_i and s_j overlap in time if they are processed concurrently at some point during the execution of* HOGWILD!*. The time during which a hyperedge s_i is being processed begins when the first coordinate in $\hat{\mathbf{x}}_i$ is read and ends after the last coordinate of $\mathbf{g}(\hat{\mathbf{x}}_i, s_i)$ is written in shared memory. We assume that there exists a number $\tau \geq 0$, such that the maximum number of sampled hyperedges that can overlap in time with a single sampled hyperedge cannot be more than τ.*

[4] throughout this section we denote $\mathbf{g}(\mathbf{x}, s_j) = \nabla f_{s_j}(\mathbf{x})$, which we assume to be bounded $\|\mathbf{g}(\mathbf{x}, s)\| \leq M$

To the best of our knowledge, all previous theoretical results on asynchronous optimization assume a bounded delay τ. If τ is small, then we expect the distance between the "fake iterate" $\mathbf{x}_j$ and the noisy iterate $\hat{\mathbf{x}}_j$ to be small. Observe that $\tau = 0$ implies that the iterates are completely decoupled, for which we obtain the exact same iterative formula of serial SGM.

To measure the distance between $\hat{\mathbf{x}}_j$ and $\mathbf{x}_j$, observe that any difference between them is solely caused by hyperedges that overlap with s_j in time. To see this, let s_i be an "earlier" sample, *i.e.* $i < j$, that does not overlap with s_j in time. This implies that the processing of s_i finishes before s_j starts being processed. Hence, the full contribution of $\gamma \mathbf{g}(\hat{\mathbf{x}}_i, s_i)$ will be recorded in both $\hat{\mathbf{x}}_j$ and $\mathbf{x}_j$ (for the latter this holds by definition). Similarly, if $i > j$ and s_i does not overlap with s_j in time, then neither $\hat{\mathbf{x}}_j$ nor $\mathbf{x}_j$ (for the latter, again by definition) contain *any* of the coordinate updates involved in the gradient update $\gamma \mathbf{g}(\hat{\mathbf{x}}_i, s_i)$. Assumption 3 guarantees that if $i < j - \tau$ or $i > j + \tau$, then the sample s_i *does not* overlap in time with s_j.

By the above discussion, and due to Assumption 3, there exist diagonal matrices $\mathbf{S}_i^j$ with diagonal entries in $\{-1, 0, 1\}$ such that

$$\hat{\mathbf{x}}_j - \mathbf{x}_j = \sum_{\substack{i=j-\tau \\ i \neq j}}^{j+\tau} \gamma \mathbf{S}_i^j \mathbf{g}(\hat{\mathbf{x}}_i, s_i). \tag{3.4}$$

These diagonal matrices account for any possible pattern of (potentially) partial updates that can occur while hyperedge s_j is being processed. We would like to note that the above notation bares resemblance to the coordinate update mismatch of asynchronous coordinate based algorithms, as in [21, 28, 29].

We now bound our error terms R_1^j and R_2^j; R_0^j is already assumed to be at most M^2 (a standard assumption even in the serial SGM literature).

Lemma 2. HOGWILD! *satisfies recursion* (2.8) *with*

$$R_1^j = \mathbb{E}\|\hat{\mathbf{x}}_j - \mathbf{x}_j\|^2 \leq \mathcal{O}(1) \cdot \gamma^2 M^2 \left(\tau + \tau^2 \frac{\overline{\Delta}_C}{n} \right) \tag{3.5}$$

$$R_2^j = \mathbb{E}\left\langle \hat{\mathbf{x}}_j - \mathbf{x}_j, \mathbf{g}(\hat{\mathbf{x}}_j, s_j) \right\rangle \leq \mathcal{O}(1) \cdot \gamma M^2 \cdot \tau \frac{\overline{\Delta}_C}{n} \tag{3.6}$$

Proof. Observe that we can bound the norm of the mismatch in the following way

$$\begin{aligned} R_1^j = \gamma^2 \mathbb{E}\left\| \sum_{\substack{i=j-\tau \\ i \neq j}}^{j+\tau} \mathbf{S}_i^j \mathbf{g}(\hat{\mathbf{x}}_i, s_i) \right\|^2 &\leq \gamma^2 \sum_i \mathbb{E}\|\mathbf{S}_i^j \mathbf{g}(\hat{\mathbf{x}}_i, s_i)\|^2 + \gamma^2 \sum_{\substack{i,k \\ i \neq k}} \mathbb{E}\left| \left\langle \mathbf{S}_i^j \mathbf{g}(\hat{\mathbf{x}}_i, s_i), \mathbf{S}_k^j \mathbf{g}(\hat{\mathbf{x}}_k, s_k) \right\rangle \right| \\ &\leq \gamma^2 \sum_i \mathbb{E}\|\mathbf{g}(\hat{\mathbf{x}}_i, s_i)\|^2 + \gamma^2 \sum_{\substack{i,k \\ i \neq k}} \mathbb{E}\left\{ \|\mathbf{g}(\hat{\mathbf{x}}_i, s_i)\| \|\mathbf{g}(\hat{\mathbf{x}}_k, s_k)\| \mathbb{1}(s_i \cap s_k \neq \emptyset) \right\}. \end{aligned}$$

since $\mathbf{S}_i^j$ are diagonal sign matrices and since the steps $\mathbf{g}(\hat{\mathbf{x}}_i, s_i)$ are supported on the samples s_i. We upper bound $\|\mathbf{g}(\hat{\mathbf{x}}_i, s_i)\| \leq M$ to obtain

$$R_1^j \leq \gamma M^2 (2\tau) + \gamma^2 M^2 (2\tau)^2 \Pr(s_i \cap s_j \neq \emptyset) \leq \mathcal{O}(1) \gamma^2 M^2 \left(\tau + \tau^2 \frac{\overline{\Delta}_{\mathrm{C}}}{n} \right).$$

where the last step follows because two randomly sampled hyperedges (with replacement) intersect with probability at most $2\frac{\overline{\Delta}_{\mathrm{C}}}{n}$, where $\overline{\Delta}_{\mathrm{C}}$ is the average degree of the conflict graph between the hyperedges.

We can bound R_2^j in a similar way:

$$R_2^j = \gamma \sum_{\substack{i=j-\tau \\ i \neq j}}^{j+\tau} \mathbb{E}\left\langle \mathbf{S}_i^j \mathbf{g}(\hat{\mathbf{x}}_i, s_i), \mathbf{g}(\hat{\mathbf{x}}_j, s_j) \right\rangle \leq \gamma M^2 \sum_{\substack{i=j-\tau \\ i \neq j}}^{j+\tau} \mathbb{E}\left\{ \mathbb{1}(s_i \cap s_j \neq \emptyset) \right\} \leq \mathcal{O}(1) \cdot \gamma M^2 \tau \frac{\overline{\Delta}_{\mathrm{C}}}{n}.$$

□

Plugging the bounds of Lemma 2 in our recursive formula, asserts that HOGWILD! satisfies the recursion

$$a_{j+1} = (1 - \gamma m)\, a_j + \mathcal{O}(1)\gamma^2 M^2 \left(1 + \tau \frac{\overline{\Delta}_{\mathrm{C}}}{n} + \gamma\tau + \gamma\tau^2 \frac{\overline{\Delta}_{\mathrm{C}}}{n}\right). \tag{3.7}$$

At this point we are done. Observe that serial SGM satisfies the recursion

$$a_{j+1} \leq (1 - \gamma m)\, a_j + \gamma^2 M^2.$$

In this case, if we choose the step-size to be $\gamma = \frac{\epsilon m}{2M^2}$, then we attain the target accuracy ϵ in $T \geq \mathcal{O}(1)M^2/(\epsilon m^2) \log\left(\frac{a_0}{\epsilon}\right)$ iterations. If τ—a proxy for the number of cores—is $\mathcal{O}(\min\{n/\overline{\Delta}_{\mathrm{C}}, M^2/(\epsilon m^2)\})$, then HOGWILD! satisfies the recursion

$$a_{j+1} \leq (1 - \gamma m) a_j + \mathcal{O}(1)\gamma^2 M^2.$$

This directly implies our main result for this section.

Theorem 3. *If the number of samples that overlap in time with a single sample during the execution of* HOGWILD! *is bounded as*

$$\tau = \mathcal{O}\left(\min\left\{\frac{n}{\overline{\Delta}_C}, \frac{M^2}{\epsilon m^2}\right\}\right),$$

then HOGWILD! *with step-size* $\gamma = \mathcal{O}(1)\frac{\epsilon m}{M^2}$*, reaches an accuracy of* $\mathbb{E}\|\mathbf{x}_k - \mathbf{x}^*\|^2 \leq \epsilon$ *after*

$$T \geq \mathcal{O}(1)\frac{M^2 \log\left(\frac{a_0}{\epsilon}\right)}{\epsilon m^2}$$

iterations.

The above iteration bound is up to a constant factor the same as the bound for serial SGM. This implies a linear speedup.[5] Observe that although the $\frac{M^2}{\epsilon m^2}$ bound on τ might seem restrictive, it is—up to a logarithmic factor—proportional to the total number of iterations required by HOGWILD! (or even serial SGM) to reach ϵ accuracy. Assuming that the conflict graph average degree $\overline{\Delta}_{\mathrm{C}}$ is constant, and that we perform a pass over the data, *i.e.*, $T = n$, then τ can be up to $\tilde{\mathcal{O}}(n)$, *i.e.*, nearly-linear in the number of function terms.[6]

Comparison with the original Hogwild! analysis of [1] We would like to summarize the key points of improvement compared to the original HOGWILD! analysis:

- Our analysis is elementary and compact, and follows simply by bounding the R_0^j, R_1^j, R_2^j terms, after introducing the perturbed gradient framework of Section 2.
- We do not assume consistent reads: while a core is reading the shared variable other cores are allowed to read, or write.
- In [1] the authors analyze a version of HOGWILD! where for each sampled hyperedge only a randomly selected coordinate is updated. Here we analyze the "full-update" version of HOGWILD!.
- The previous work of [1] establishes a nearly-linear speedup for HOGWILD! if τ, the proxy for the number of cores, is bounded as

  $$\tau = \mathcal{O}\left(\sqrt[4]{\frac{n}{\Delta_{\mathrm{R}} \cdot \Delta_{\mathrm{L}}^2}}\right)$$

  where Δ_{R} is the maximum right degree of the term-variables bipartite graph, shown in Fig 1, and Δ_{L} is the maximum left degree of the same graph. Observe that $\Delta_{\mathrm{R}} \cdot \Delta_{\mathrm{L}}^2 \geq \Delta_{\mathrm{L}} \cdot \Delta_{\mathrm{C}}$, where Δ_{C} is the maximum degree of the function term conflict graph. Here, we obtain a linear speedup for up to

  $$\tau = \mathcal{O}\left(\min\left\{\frac{n}{\overline{\Delta}_{\mathrm{C}}}, \frac{M^2}{\epsilon m^2}\right\}\right)$$

  where $\overline{\Delta}_{\mathrm{C}}$ is only the average degree of the conflict graph in Fig 1.

[5] We would like to note that a rate of $O(1/\epsilon)$ can be obtained using the epoch version of SGM (see *e.g.*, [1, 31]).
[6] $\tilde{\mathcal{O}}$ hides logarithmic terms.

4 Asynchronous Stochastic Coordinate Descent

In this section, we use our perturbed gradient framework to analyze the convergence performance of asynchronous parallel stochastic coordinate descent (ASCD). This algorithm has been previously analyzed in [5, 21]. The present analysis fits in the same framework introduced above, and therefore it is compact, elementary, and requires only the assumptions made for HOGWILD!.

Algorithm 2 ASCD

1: **while** iterations $\leq T$ **do in parallel**
2: $\quad\hat{\mathbf{x}} =$ an inconsistent read of the shared variable $\mathbf{x}$
3: $\quad$Sample a coordinate s
4: $\quad u_s = -\gamma \cdot d[\nabla f(\mathbf{x})]_s$
5: $\quad[\mathbf{x}]_s = [\mathbf{x}]_s + u_s$ // atomic write
6: **end while**

ASCD, shown in Alg. 2, is a linearly convergent algorithm for minimizing strongly convex functions f. At each iteration a core samples one of the coordinates, computes a full gradient update for that coordinate, and proceeds with updating a single element of the shared memory variable $\mathbf{x}$. The challenge in analyzing ASCD, especially compared to HOGWILD!, is that we not only need to bound the error due to the asynchrony between cores, but we also show that this error decays fast, as we are getting closer to the optimal solution. However, we will show how our simple perturbed iterate framework can handle this type of noise analysis in a straightforward manner, using simple recursive bounds.

We define $\hat{\mathbf{x}}_i$ as in the previous section, but now the samples s_i are coordinates sampled uniformly at random from $\{1, 2, \ldots, d\}$. After T samples have been processed completely, the following vector is contained in shared memory:

$$\underbrace{\overbrace{\mathbf{x}_0 - \gamma d[\nabla f(\hat{\mathbf{x}}_0)]_{s_0}\mathbf{e}_{s_0}}^{\mathbf{x}_1} - \ldots - \gamma d[\nabla f(\hat{\mathbf{x}}_{T-1})]_{s_{T-1}}\mathbf{e}_{s_{T-1}}}_{\mathbf{x}_T}, \tag{4.1}$$

where $\mathbf{x}_0$ is the initial guess, $\mathbf{e}_{s_j}$ is the standard basis vector with a one at position s_j, $[\nabla f(\mathbf{x})]_{s_j}$ denotes the s_j-th coordinate of the gradient of f computed on $\mathbf{x}$. Hence, similar to HOGWILD! in the previous section, ASCD satisfies the following iterative formula

$$\mathbf{x}_{j+1} = \mathbf{x}_j - \gamma \cdot d \cdot [\nabla f(\hat{\mathbf{x}}_j)]_{s_j}\mathbf{e}_{s_j} = \mathbf{x}_j - \gamma \cdot \mathbf{g}(\hat{\mathbf{x}}_j, s_j)$$

Notice that $\mathbb{E}_{s_j}\mathbf{g}(\hat{\mathbf{x}}_j, s_j) = \nabla f(\hat{\mathbf{x}}_j)$, and thus, similarly to HOGWILD!, ASCD's iterates $a_j = \mathbb{E}\|\mathbf{x}_j - \mathbf{x}^*\|^2$ satisfy the recursion of Eq. (2.8):

$$a_{j+1} \leq (1 - \gamma m)\, a_j + \gamma^2 \underbrace{\mathbb{E}\|g(\hat{\mathbf{x}}_j, s_j)\|^2}_{R_0^j} + 2\gamma m \underbrace{\mathbb{E}\|\hat{\mathbf{x}}_j - \mathbf{x}_j\|^2}_{R_1^j} + 2\gamma \underbrace{\mathbb{E}\left\langle \hat{\mathbf{x}}_j - \mathbf{x}_j, \mathbf{g}(\hat{\mathbf{x}}_j, s_j)\right\rangle}_{R_2^j}. \tag{4.2}$$

Before, stating our main result, let us introduce some further notation. Let us define the largest distance between the optimal vector, and what is read by the cores during the execution of the algorithm:

$$\hat{a}_0 \coloneqq \max_{0 \leq k \leq T} \mathbb{E}\|\hat{\mathbf{x}}_k - \mathbf{x}^*\|^2 \tag{4.3}$$

which should be thought of as proportional to $a_0 = \mathbb{E}\|\mathbf{x}_0 - \mathbf{x}^*\|^2$. Furthermore, by a simple application of the L-Lipschitz assumption on f, we have a uniform bound on the norm of each computed gradient

$$M^2 \coloneqq \max_{0 \leq k \leq T} \mathbb{E}\|\nabla f(\hat{\mathbf{x}}_k)\|^2 \leq L^2 \hat{a}_0. \tag{4.4}$$

Here we assume that the optimization takes place in a ℓ_∞ ball, so that $M < \infty$. This simply means that the iterates will never have infinitely large coordinate values. This assumption is made in previous work

explicitly or implicitly, and in practice it can be implemented easily since the projection on a ℓ_∞ ball can be done component wise. Finally, let us define the condition number of f as

$$\kappa := \frac{L}{m}$$

where L is the Lipschitz constant, and m the strong convexity parameter. We are now ready to state our main result for this section.

Theorem 4. *Let the maximum number of coordinates that can be concurrently processed while a core is processing a single coordinate be at most*

$$\tau = \mathcal{O}\left(\min\left\{\frac{\kappa\sqrt{d}}{\log\left(\frac{\hat{a}_0}{\epsilon}\right)}, \sqrt[6]{d}\right\}\right).$$

Then, ASCD with step-size $\gamma = \frac{\mathcal{O}(1)}{dL\kappa}$ achieves an error $\mathbb{E}\|\mathbf{x}_k - \mathbf{x}^*\|^2 \leq \epsilon$ after

$$k \geq \mathcal{O}(1) \cdot d\kappa^2 \log\left(\frac{a_0}{\epsilon}\right)$$

iterations.

Observe that serial SCD, using the recursive formula in (2.8), and the same step-size of the above theorem, can be shown to achieve the same accuracy as ASCD in order-wise the same number of steps. Hence, as long as the proxy for the number of cores is bounded as

$$\tau = \mathcal{O}\left(\min\left\{\frac{\kappa\sqrt{d}}{\log\left(\frac{\hat{a}_0}{\epsilon}\right)}, \sqrt[6]{d}\right\}\right)$$

our theorem implies a linear speedup with respect to this simple convergence bound. We would like to note however, that relevant coordinate descent literature sometimes uses more refined properties on the function at hand, that can lead to potentially better convergence bounds, especially in terms of function value accuracy, i.e., $f(\mathbf{x}_k) - f(\mathbf{x}^*)$ (see e.g., [5, 21, 32]).

We would further like to remark that between the two bounds on τ, the second one, i.e. $\mathcal{O}(\sqrt[6]{d})$, is the more restrictive, as the first one is proportional—up to log factors—to the square root of the number of iterations, which is usually $\Omega(d)$. We explain in our subsequent derivation how this loose bound can be improved, but leave it as an open question for future work.

4.1 Proof of Theorem 4

The analysis here is slightly more involved compared to HOGWILD!, yet it is still compact. The main technical bottleneck is to relate the decay of R_0^j with that of R_1^j, and then to exploit the sparsity of the updates for bounding R_2^j.

We start with a simple upper bound on the norm of the gradient updates. From the L-Lipschitz assumption on $\nabla f(\mathbf{x})$, we have

$$\mathbb{E}_{s_k}\left\|\mathbf{g}(\hat{\mathbf{x}}_k, s_k)\right\|^2 = d \cdot \|\nabla f(\hat{\mathbf{x}}_k)\|^2 \leq dL^2\|\hat{\mathbf{x}}_k - \mathbf{x}^*\|^2 \leq 2dL^2\|\mathbf{x}_j - \mathbf{x}^*\|^2 + 2dL^2\|\mathbf{x}_j - \hat{\mathbf{x}}_k\|^2$$

where the last inequality is due to Jensen's Inequality. This yields the following result.

Lemma 5. *For any k and j we have $\mathbb{E}\left\|\mathbf{g}(\hat{\mathbf{x}}_k, s_k)\right\|^2 \leq 2dL^2\left(a_j + \mathbb{E}\|\mathbf{x}_j - \hat{\mathbf{x}}_k\|^2\right)$.*

Let T be the total number iterations for which ASCD runs and let us define the set

$$\mathcal{S}_r^j = \{\max\{j - r\tau, 0\}, \ldots, j-1, j, j+1, \ldots, \min\{j + r\tau, T\}\}$$

which has cardinality at most $2r\tau + 1$ and contains all indices around j within $r\tau$ steps, as sketched in Fig. 2.

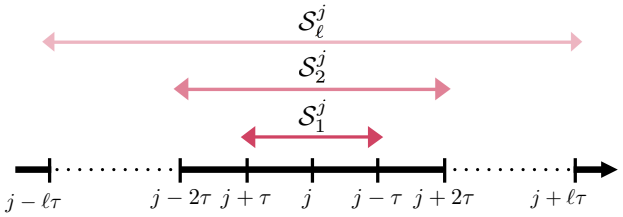


Figure 2: The set $\mathcal{S}_r^j = \{\max\{j - r\tau, 0\}, \ldots, j-1, j, j+1, \ldots, \min\{j + r\tau, T\}\}$ comprises the indices around j (including j) within $r\tau$ steps. The cardinality of such set is $2r\tau + 1$. Here, $\mathcal{S}_0^j = \{j\}$.

Due to Assumption 3, and similar to [21], there exist $\sigma_{i,k}^j \in \{-1, 0, 1\}$ variables such that, for any index k in the set $\mathcal{S}_r^j$, we have

$$\hat{\mathbf{x}}_k - \mathbf{x}_j = \sum_{i \in \mathcal{S}_{r+1}^j} \sigma_{i,k}^j \cdot \gamma \mathbf{g}(\hat{\mathbf{x}}_i, s_i). \tag{4.5}$$

The above equation implies that the difference between a "fake" iterate at time j and what was read at time k, can be expressed as a linear combination of any coordinate updates that occurred during the time interval defined by $\mathcal{S}_{r+1}^j$.

From Eq. (4.5) we see that $\|\hat{\mathbf{x}}_k - \mathbf{x}_j\|$, for any $k \in \mathcal{S}_r^j$, can be upper bounded in terms of the magnitude of the coordinate updates that occurred in $\mathcal{S}_{r+1}^j$; since these coordinate updates are coordinates of the true gradient, we will be able to use their norm to bound the norm of $\hat{\mathbf{x}}_k - \mathbf{x}_j$. This will be useful towards bounding R_1^j. Moreover, Lemma 5 shows that the magnitude of the gradient steps can be upper bounded in terms of the size of the mismatches. This will in turn be useful in bounding R_0^j. The above observations are fundamental to our approach. The following lemma makes the above ideas explicit.

Lemma 6. *For any $j \in \{0, \ldots, T\}$, we have*

$$\max_{k \in \mathcal{S}_r^j} \mathbb{E}\|\mathbf{g}(\hat{\mathbf{x}}_k, s_k)\|^2 \leq 2dL^2 \left(a_j + \max_{k \in \mathcal{S}_r^j} \mathbb{E}\|\hat{\mathbf{x}}_k - \mathbf{x}_j\|^2 \right) \tag{4.6}$$

$$\max_{k \in \mathcal{S}_r^j} \mathbb{E}\|\hat{\mathbf{x}}_k - \mathbf{x}_j\|^2 \leq (3\gamma\tau(r+1))^2 \max_{k \in \mathcal{S}_{r+1}^j} \mathbb{E}\|\mathbf{g}(\hat{\mathbf{x}}_k, s_k)\|. \tag{4.7}$$

Proof. The first inequality is a trivial consequence of Lemma 5. For the second, let $k \in \mathcal{S}_r^j$. Then, as we mentioned previously, we have $\hat{\mathbf{x}}_k - \mathbf{x}_j = \sum_{i \in \mathcal{S}_{r+1}^j} \sigma_{i,k} \gamma \mathbf{g}(\hat{\mathbf{x}}_i, s_i)$. Hence,

$$\begin{aligned}
\mathbb{E}\|\hat{\mathbf{x}}_k - \mathbf{x}_j\|^2 &= \gamma^2 \cdot \mathbb{E}\left\{ \left\| \sum_{i \in \mathcal{S}_{r+1}^j} \sigma_{i,k}^j \cdot \mathbf{g}(\hat{\mathbf{x}}_i, s_i) \right\|^2 \right\} \leq \gamma^2 \cdot \mathbb{E}\left\{ |\mathcal{S}_{r+1}^j| \sum_{i \in \mathcal{S}_{r+1}^j} \|\mathbf{g}(\hat{\mathbf{x}}_i, s_i)\|^2 \right\} \\
&\leq \gamma^2 \cdot |\mathcal{S}_{r+1}^j|^2 \max_{i \in \mathcal{S}_{r+1}^j} \mathbb{E}\|\mathbf{g}(\hat{\mathbf{x}}_i, s_i)\|^2 \leq (3\gamma\tau(r+1))^2 \max_{i \in \mathcal{S}_{r+1}^j} \mathbb{E}\|\mathbf{g}(\hat{\mathbf{x}}_i, s_i)\|^2,
\end{aligned}$$

where the first inequality follows due to Jensen's inequality, and the last inequality uses the bound $|S_{r+1}^j| \leq 2(r+1)\tau + 1 \leq 3\tau(r+1)$. ∎

Let us now define for simplicity

$$G_r = \max_{k \in \mathcal{S}_r^j} \mathbb{E}\|\mathbf{g}(\hat{\mathbf{x}}_k, s_k)\|^2 \quad \text{and} \quad \Delta_r = \max_{k \in \mathcal{S}_r^j} \mathbb{E}\|\hat{\mathbf{x}}_k - \mathbf{x}_j\|^2.$$

We further note that that all gradient norms can be bounded as

$$G_r = \max_{k \in \mathcal{S}_r^j} \mathbb{E}\|\mathbf{g}(\hat{\mathbf{x}}_k, s_k)\|^2 = d \max_{k \in \mathcal{S}_r^j} \mathbb{E}\|\nabla f(\hat{\mathbf{x}}_k)\|^2 \leq d \max_{0 \leq k \leq T} \mathbb{E}\|\nabla f(\hat{\mathbf{x}}_k)\|^2 = dM^2,$$

a property that we will use in our bounds. Observe that $R_0^j = \mathbb{E}\|\mathbf{g}(\hat{\mathbf{x}}_j, s_j)\|^2 = G_0$ and $R_1^j = \mathbb{E}\|\hat{\mathbf{x}}_j - \mathbf{x}_j\|^2 = \Delta_0$. To obtain bounds for our first two error terms, R_0^j and R_1^j, we will expand the recursive relations that are implied by Lemma 6. By expanding the above recursive formulas, we establish in Section A.1 of the Appendix the following bounds.

Lemma 7. *Let $\tau \leq \frac{\kappa\sqrt{d}}{\ell}$ and set $\gamma = \frac{\theta}{6dL\kappa}$ for any fixed $\theta \leq 1$ and $\ell \geq 1$. Then,*

$$R_0^j \leq \mathcal{O}(1)\left(dL^2 a_j + \theta^{2\ell} dM^2\right) \text{ and } R_1^j \leq \mathcal{O}(1)\left(\theta^2 a_j + \theta^{2\ell}\frac{M^2}{L^2}\right).$$

Observe that by the Cauchy-Schwartz inequality we can immediately obtain the bound $R_2^j \leq \sqrt{R_0^j R_1^j}$. Unfortunately this approach yields a result that can only guarantee convergence up a factor of $\sqrt{d}$ slower than serial SCD. This happens because upper bounding the inner product $\langle \hat{\mathbf{x}}_j - \mathbf{x}_j, \mathbf{g}(\hat{\mathbf{x}}_j, s_j)\rangle$ by $\|\hat{\mathbf{x}}_j - \mathbf{x}_j\|\|\mathbf{g}(\hat{\mathbf{x}}_j, s_j)\|$ disregards the extreme sparsity of $\mathbf{g}(\hat{\mathbf{x}}_j, s_j)$. The next lemma uses slightly more involved argument to bound R_2^j that exploits the sparsity of the gradient update. The proof can be found in Appendix A.1

Lemma 8. *Assume $\tau \leq \frac{\kappa\sqrt{d}}{\ell}$ and $\tau = \mathcal{O}(\sqrt[6]{d})$. Then,*

$$R_2^j \leq \mathcal{O}(1)\left(\theta \cdot m \cdot a_j + \theta^{2\ell}\frac{M^2}{L\kappa}\right).$$

Remark 1. *We note that the τ^2 factor in the upper bound on $\max \mathbb{E}\|\hat{\mathbf{x}}_k - \mathbf{x}_j\|^2$ in Lemma 6 can be loose. We believe that it should instead be τ, when τ is smaller than some measure of the sparsity. If the sparsity of the steps $\mathbf{g}(\hat{\mathbf{x}}_i, s_i)$ can be exploited, the condition $\tau = \mathcal{O}(\sqrt[6]{d})$ in Theorem 4 could be improved to $\tau = \mathcal{O}(\sqrt[4]{d})$.*

4.1.1 Putting it all together

We can now plug in the upper bounds on R_0^j, R_1^j, and R_2^j in our perturbed iterate recursive formula

$$a_{j+1} \leq (1-\gamma m)\, a_j + \gamma^2 \underbrace{\mathbb{E}\|g(\hat{\mathbf{x}}_j, s_j)\|^2}_{R_0^j} + 2\gamma m \underbrace{\mathbb{E}\|\hat{\mathbf{x}}_j - \mathbf{x}_j\|^2}_{R_1^j} + 2\gamma \underbrace{\mathbb{E}\left\langle \hat{\mathbf{x}}_j - \mathbf{x}_j, \mathbf{g}(\hat{\mathbf{x}}_j, s_j)\right\rangle}_{R_2^j}$$

to find that ASCD satisfies

$$a_{j+1} \leq \left(1 - \gamma m + \underbrace{\mathcal{O}(1)\left(\gamma^2 dL^2 + \gamma m\theta^2 + \gamma\theta m\right)}_{=r(\gamma)}\right) a_j + \mathcal{O}(1)\left(\underbrace{\gamma^2\theta^{2\ell} dM^2 + \gamma\theta^{2\ell}\frac{M^2}{L\kappa}}_{=\delta(\gamma)}\right).$$

Observe that in the serial case of SCD the errors R_1^j and R_2^j are zero, and $R_0^j = \mathbb{E}\|g(\mathbf{x}_j, s_j)\|^2$. By applying the Lipschitz assumption on f, we get $\mathbb{E}\|g(\mathbf{x}_j, s_j)\|^2 \leq dL^2 a_j$, and obtain the recursive formula

$$a_{j+1} \leq (1 - \gamma m + \gamma^2 dL^2) a_j. \tag{4.8}$$

If we hope for ASCD to follow the same recursion, i.e., if we want it to have order-wise the same convergence rate as the serial recursive bound implied by Eq. (4.8), then we require that $\gamma m - r(\gamma) \geq C(\gamma m - \gamma^2 dL^2)$, where $C < 1$ is a constant. Solving for γ we get

$$\begin{aligned}
&\gamma m - C'\left(\gamma^2 dL^2 + \gamma m\theta^2 + \gamma\theta m\right) \geq C(\gamma m - \gamma^2 dL^2) \\
\Leftrightarrow &(1-C)\gamma m - (C'-C)\gamma^2 dL^2 + C'(\gamma m\theta^2 + \gamma\theta m) \geq 0 \\
\Leftrightarrow &(C'-C)\gamma dL^2 \leq [(1-C) + C'(\theta^2+\theta)]m \\
\Leftrightarrow &\gamma \leq \mathcal{O}(1)\frac{\theta m}{dL^2} = \mathcal{O}(1)\frac{\theta}{d\kappa L}
\end{aligned}$$

where $C' > 1$ is some absolute constant. For $\gamma = \mathcal{O}(1)\frac{\theta}{d\kappa L}$, the $\delta(\gamma)$ term in the recursive bound becomes

$$\mathcal{O}(1)\left(\frac{\theta^2}{d^2\kappa^2L^2}\theta^{2\ell}dM^2 + \frac{\theta}{d\kappa L}\theta^{2\ell}\frac{M^2}{L\kappa}\right) = \mathcal{O}(1)\left(\theta^{2\ell+2}\frac{M^2}{d\kappa^2L^2} + \theta^{2\ell+1}\frac{M^2}{d\kappa^2L^2}\right) \leq \mathcal{O}(1)\theta^{2\ell}\frac{\hat{a}_0}{d\kappa^2}$$

where we used the inequality $M^2 \leq L^2\hat{a_0}$ from Eq. (4.4). Hence, ASCD satisfies the recursion

$$a_{j+1} \leq \left(1 - \mathcal{O}(1)\frac{\theta}{d\kappa^2}\right)a_j + \mathcal{O}(1)\theta^{2\ell}\frac{\hat{a}_0}{d\kappa^2} \leq \left(1 - \mathcal{O}(1)\frac{\theta}{d\kappa^2}\right)^{j+1}a_0 + \mathcal{O}(1)\theta^{2\ell}\hat{a}_0,$$

Let us set θ to be a sufficiently small constant so that $\mathcal{O}(1)\frac{\theta}{d\kappa^2} = \frac{1}{d\kappa^2}$ and solve for ℓ such that $\mathcal{O}(1)\theta^{2\ell}\hat{a}_0 = \epsilon/2$. This gives $\ell = \mathcal{O}(1)\log\left(\frac{\hat{a}_0}{\epsilon}\right)$. The main result now follows from solving $\left(1 - \frac{\mathcal{O}(1)}{d\kappa^2}\right)^{j+1}a_0 = \epsilon/2$ for j.

5 Sparse and Asynchronous SVRG

5.1 Serial Sparse SVRG

In this section, we introduce a sparse version of the SVRG algorithm that was presented in [11]. We proceed with introducing and analyzing KroMagnon, a parallel, asynchronous and sparse variant of SVRG. We observe that we can derive the error bounds for KroMagnon in a nearly identical way to ASCD, with minor modifications.

SVRG runs for a number of epochs; the per epoch iteration is given below

$$\mathbf{x}_{j+1} = \mathbf{x}_j - \gamma\left(\mathbf{g}(\mathbf{x}_j, s_j) - \mathbf{g}(\mathbf{y}, s_j) + \nabla f(\mathbf{y})\right) \tag{5.1}$$

where $\mathbf{y}$ is the last iterate of the previous epoch, and as such is updated at the end of every epoch. Here f is of the same form as Section 3:

$$f(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n} f_{e_i}(\mathbf{x})$$

and $\mathbf{g}(\mathbf{x}, s_j) = \nabla f_{s_j}(\mathbf{x})$, with $s_j \in \mathcal{E}$ hyperedges sampled uniformly at random. As is common in the SVRG literature, we further assume that the individual f_{e_i} terms are L-smooth. The theoretical innovation in SVRG is having an SGM flavored algorithm, with small amortized cost per iteration, where the variance of the gradient estimate is smaller than that of standard SGM. For a certain selection of learning rate, epoch size, and number of iterations, [11] establishes that SVRG attains a linear rate.

Observe that when optimizing a decomposable f with sparse terms, and in contrast to SGM, the SVRG iterates will be dense due to the term $\nabla f(\mathbf{y})$. From a practical perspective, when the SGM iterates are very sparse—the case in several applications [1]—the cost of writing a sparse update in shared memory is significantly smaller than applying the dense gradient update term $\nabla f(\mathbf{y})$. Furthermore, these dense updates will cause significantly more memory conflicts in an asynchronous execution, amplifying the error terms in (2.8), and introducing time delays due to memory contention.

We introduce *sparse SVRG* where the support of the update is determined by that of $\mathbf{g}(\mathbf{y}, s_j)$:

$$\mathbf{x}_{j+1} = \mathbf{x}_j - \gamma\left(\mathbf{g}(\mathbf{x}_j, s_j) - \mathbf{g}(\mathbf{y}, s_j) + \mathbf{D}_{s_j}\nabla f(\mathbf{y})\right) = \mathbf{x}_j - \gamma\mathbf{v}_j,$$

where $\mathbf{D}_{s_j} = \mathbf{P}_{s_j}\mathbf{D}$, and $\mathbf{P}_{s_j}$ is the projection on the support of s_j and $\mathbf{D} = \operatorname{diag}\left(p_1^{-1}, \ldots, p_d^{-1}\right)$ is a $d \times d$ diagonal matrix. The weight p_v is equal to the probability that index v belongs to a hyperedge sampled uniformly at random from $\mathcal{E}$. These probabilities can be simply computed by the normalized right degrees of the bipartite term-variable graph shown in Fig. 1. The above normalization ensures that $\mathbb{E}_{s_j}\mathbf{D}_{s_j}\nabla f(\mathbf{y}) = \nabla f(\mathbf{y})$ and thus that $\mathbb{E}\mathbf{v}_j = \nabla f(\mathbf{x}_j)$. In the following we establish an upper bound on $\mathbb{E}\|\mathbf{v}_j\|^2$ for sparse SVRG. This upper bound will be the same as the one used in [11] to establish a linear rate of convergence for dense SVRG.

Throughout this section we assume that there exist an $M > 0$ such that $\|\mathbf{v}_j\| \leq M$.

Lemma 9. *The variance of serial sparse SVRG satisfies*

$$\mathbb{E}\|\mathbf{v}_j\|^2 \leq 2\mathbb{E}\|\mathbf{g}(\mathbf{x}_j, s_j) - \mathbf{g}(\mathbf{x}^*, s_j)\|^2 + 2\mathbb{E}\|\mathbf{g}(\mathbf{y}, s_j) - \mathbf{g}(\mathbf{x}^*, s_j)\|^2 - 2\nabla f(\mathbf{y})^\top\mathbf{D}\nabla f(\mathbf{y}).$$

Proof. By definition $\mathbf{v}_j = \mathbf{g}(\mathbf{x}_j, s_j) - \mathbf{g}(\mathbf{y}, s_j) + \mathbf{D}_{s_j}\nabla f(\mathbf{y})$. Therefore

$$\begin{aligned}\mathbb{E}\|\mathbf{v}_j\|^2 &= \mathbb{E}\|\mathbf{g}(\mathbf{x}_j, s_j) - \mathbf{g}(\mathbf{y}, s_j) + \mathbf{D}_{s_j}\nabla f(\mathbf{y})\|^2 \\ &\leq 2\mathbb{E}\|\mathbf{g}(\mathbf{x}_j, s_j) - \mathbf{g}(\mathbf{x}^*, s_j)\|^2 + 2\mathbb{E}\|\mathbf{g}(\mathbf{y}, s_j) - \mathbf{g}(\mathbf{x}^*, s_j) - \mathbf{D}_{s_j}\nabla f(\mathbf{y})\|^2.\end{aligned}$$

We expand the second term to find that

$$\begin{aligned}&\mathbb{E}\|\mathbf{g}(\mathbf{y}, s_j) - \mathbf{g}(\mathbf{x}^*, s_j) - \mathbf{D}_{s_j}\nabla f(\mathbf{y})\|^2 \\ &= \mathbb{E}\|\mathbf{g}(\mathbf{y}, s_j) - \mathbf{g}(\mathbf{x}^*, s_j)\|^2 - 2\mathbb{E}\left\langle \mathbf{g}(\mathbf{y}, s_j) - \mathbf{g}(\mathbf{x}^*, s_j), \mathbf{D}_{s_j}\nabla f(\mathbf{y})\right\rangle + \mathbb{E}\|\mathbf{D}_{s_j}\nabla f(\mathbf{y})\|^2\end{aligned}$$

Since $\mathbf{g}(\mathbf{x}, s_j)$ is supported on s_j for all $\mathbf{x}$, we have

$$\begin{aligned}\mathbb{E}\left\langle \mathbf{g}(\mathbf{y}, s_j) - \mathbf{g}(\mathbf{x}^*, s_j), \mathbf{D}_{s_j}\nabla f(\mathbf{y})\right\rangle &= \mathbb{E}\left\langle \mathbf{g}(\mathbf{y}, s_j) - \mathbf{g}(\mathbf{x}^*, s_j), \mathbf{D}\nabla f(\mathbf{y})\right\rangle \\ &= \nabla f(\mathbf{y})^\top \mathbf{D}\nabla f(\mathbf{y}),\end{aligned}$$

where the second equality follows by the property of iterated expectations. The conclusion follows because $\mathbb{E}\|\mathbf{D}_{s_j}\nabla f(\mathbf{y})\|^2 = \nabla f(\mathbf{y})^\top \mathbf{D}\nabla f(\mathbf{y})$. ☐

Observe that the last term in the variance bound is a non-negative quadratic form, hence, we can drop it, and obtain the same variance bound as the one obtained in [11] for dense SVRG.

Corollary 10. *Sparse SVRG admits the same convergence rate upper bound as that of the SVRG of [11].*

We note that usually the convergence rates for SVRG are obtained for function value differences. However, since our perturbed iterate framework of Section 2 is based on iterate differences, we re-derive a convergence bound for iterates.

Lemma 11. *Let the step-size be $\gamma = \frac{1}{4L\kappa}$ and the length of an epoch be $8\kappa^2$. Then,*

$$\mathbb{E}\|\mathbf{y}_k - \mathbf{x}^*\|^2 \leq 0.75^k \cdot \mathbb{E}\|\mathbf{y}_0 - \mathbf{x}^*\|^2 \tag{5.2}$$

where $\mathbf{y}_k$ is the iterate at the end of the k-th epoch.

Proof. We bound the distance to the optimum after one epoch of length $8\kappa^2$.

$$\begin{aligned}\mathbb{E}\|\mathbf{x}_{j+1} - \mathbf{x}^*\|^2 &= \mathbb{E}\|\mathbf{x}_j - \mathbf{x}^*\|^2 - 2\gamma\mathbb{E}\left\langle \mathbf{x}_j - \mathbf{x}^*, \mathbf{v}_j\right\rangle + \gamma^2\mathbb{E}\|\mathbf{v}_j\|^2 \\ &\leq \mathbb{E}\|\mathbf{x}_j - \mathbf{x}^*\|^2 - 2\gamma\mathbb{E}\left\langle \mathbf{x}_j - \mathbf{x}^*, \nabla f(\mathbf{x}_j)\right\rangle + 2\gamma^2\mathbb{E}\|\mathbf{g}(\mathbf{x}_j, s_j) - \mathbf{g}(\mathbf{x}^*, s_j)\|^2 + 2\gamma^2\mathbb{E}\|\mathbf{g}(\mathbf{y}, s_j) - \mathbf{g}(\mathbf{x}^*, s_j)\|^2 \\ &\leq \mathbb{E}\|\mathbf{x}_j - \mathbf{x}^*\|^2 - 2\gamma\mathbb{E}\left\langle \mathbf{x}_j - \mathbf{x}^*, \nabla f(\mathbf{x}_j)\right\rangle + 2\gamma^2L^2\mathbb{E}\|\mathbf{x}_j - \mathbf{x}^*\|^2 + 2\gamma^2L^2\mathbb{E}\|\mathbf{y} - \mathbf{x}^*\|^2 \\ &\leq (1 - 2\gamma m + 2\gamma^2L^2)\mathbb{E}\|\mathbf{x}_j - \mathbf{x}^*\|^2 + 2\gamma^2L^2\mathbb{E}\|\mathbf{y} - \mathbf{x}^*\|^2\end{aligned}$$

The first inequality follows from Lemma 9 and an application of the property of iterated expectations to obtain $\mathbb{E}\left\langle \mathbf{x}_j - \mathbf{x}^*, \mathbf{v}_j\right\rangle = \mathbb{E}\left\langle \mathbf{x}_j - \mathbf{x}^*, \nabla f(\mathbf{x}_j)\right\rangle$. The second inequality follows from the smoothness of $\mathbf{g}(\mathbf{x}, s_j)$, and the third inequality follows since f is m-strongly convex.

We can rewrite the inequality derived above as

$$a_{j+1} \leq (1 - 2\gamma m + 2\gamma^2L^2)a_j + 2\gamma^2L^2a_0$$

because by construction $\mathbf{y} = \mathbf{x}_0$. Let $\gamma = \frac{1}{4L\kappa}$. Then, $1 - 2\gamma m + 2\gamma^2L^2 \leq 1 - \frac{1}{4\kappa^2}$ and

$$\sum_{i=0}^{j}(1 - 2\gamma m + 2\gamma^2L^2)^i \leq \sum_{i=0}^{j}\left(1 - \frac{1}{4\kappa^2}\right)^i \leq \sum_{i=0}^{\infty}\left(1 - \frac{1}{4\kappa^2}\right)^i = 4\kappa^2$$

since $\frac{1}{4\kappa^2} \leq \frac{1}{4}$. Therefore

$$\begin{aligned}a_{j+1} &\leq \left(1 - \frac{1}{4\kappa^2}\right)a_j + 2\gamma^2L^2a_0 \leq \left(1 - \frac{1}{4\kappa^2}\right)^{j+1}a_0 + \sum_{i=0}^{j-1}\left(1 - \frac{1}{4\kappa^2}\right)^i \cdot 2\gamma^2L^2a_0 \\ &\leq \left(1 - \frac{1}{4\kappa^2}\right)^{j+1}a_0 + 4\kappa^2\gamma^2L^2a_0 = \left[\left(1 - \frac{1}{4\kappa^2}\right)^{j+1} + \frac{1}{4}\right]a_0.\end{aligned}$$

Setting the length of an epoch to be $j = 2 \cdot (4\kappa^2)$ gives us $a_{j+1} \leq (1/2 + 1/4) \cdot a_0 = 0.75 \cdot a_0$, and the conclusion follows. ☐

The above lead to the following convergence rate result.

Theorem 12. *Sparse SVRG, with step-size* $\gamma = \mathcal{O}(1)\frac{1}{L\kappa}$ *and epoch size* $S = \mathcal{O}(1)\kappa^2$*, reaches accuracy* $\mathbb{E}\|\mathbf{y}_E - \mathbf{x}^*\|^2 \leq \epsilon$ *after*

$$E = \mathcal{O}(1)\log\left(\frac{a_0}{\epsilon}\right)$$

epochs, where $\mathbf{y}_E$ *is the last iterates of the final epoch, and* $a_0 = \|\mathbf{x}_0 - \mathbf{x}^*\|^2$ *is the initial distance squared to the optimum.*

5.2 KroMagnon: Asynchronous Parallel Sparse SVRG

KroMagnon is our asynchronous implementation of sparse SVRG, and is given below as Algorithm 3.

Algorithm 3 KroMagnon

x = **y** = $\mathbf{x}_0$
for Epoch = 1 : E **do**
  Compute in parallel $\mathbf{z} = \nabla f(\mathbf{y})$
  while number of sampled edges $\leq S$ **do in parallel**
    sample a random hyperedge s
    $[\hat{\mathbf{x}}]_s$ = an inconsistent read of the shared variable $[\mathbf{x}]_s$
    $[\mathbf{u}]_s = -\gamma \cdot \left(\nabla f_s([\mathbf{x}]_s) - \nabla f_s([\mathbf{y}]_s) - \mathbf{D}_{s_j}\mathbf{z}\right)$
    for $v \in s$ **do**
      $[\mathbf{x}]_v = [\mathbf{x}]_v + [\mathbf{u}]_v$ // atomic write
    end for
  end while
  y = **x**
end for

Let $\mathbf{v}(\hat{\mathbf{x}}_j, s_j) = \mathbf{g}(\hat{\mathbf{x}}_j, s_j) - \mathbf{g}(\mathbf{y}, s_j) + \mathbf{D}_{s_j}\nabla f(\mathbf{y})$ be the noisy gradient update vector. Then, the following vector is contained in the shared memory after processing a total of T hyperedges:

$$\underbrace{\overbrace{\mathbf{x}_0 - \gamma\mathbf{v}(\hat{\mathbf{x}}_0, s_0)}^{\mathbf{x}_1} - \ldots - \gamma\mathbf{v}(\hat{\mathbf{x}}_{T-1}, s_{T-1})}_{\mathbf{x}_T}. \tag{5.3}$$

We now define the perturbed iterates as

$$\mathbf{x}_{i+1} = \mathbf{x}_i - \gamma\mathbf{v}(\hat{\mathbf{x}}_i, s_i) \tag{5.4}$$

for $i = 0, 1, ..., T-1$, where s_i is the i-th uniformly sampled hyperedge. Since $\mathbb{E}\mathbf{v}(\hat{\mathbf{x}}_j, s_j) = \nabla f(\mathbf{x}_j)$, KroMagnon also satisfies the following form of recursion (2.8):

$$a_{j+1} \leq (1-\gamma m)a_j + \gamma^2 \underbrace{\mathbb{E}\|\mathbf{v}(\hat{\mathbf{x}}_j, s_j)\|^2}_{R_0^j} + 2\gamma m \underbrace{\mathbb{E}\|\hat{\mathbf{x}}_j - \mathbf{x}_j\|^2}_{R_1^j} + 2\gamma \underbrace{\mathbb{E}\left\langle \hat{\mathbf{x}}_j - \mathbf{x}_j, \mathbf{v}(\hat{\mathbf{x}}_j, s_j)\right\rangle}_{R_2^j}$$

To prove the convergence of KroMagnon we follow the same line of reasoning presented in the previous section. Most of the arguments that we use here come from a trivial generalization of the analysis of ASCD. Our main result of this section is given below.

Theorem 13. *Let the maximum number of samples that can overlap in time with a single sample be bounded as*

$$\tau = \mathcal{O}\left(\min\left\{\frac{\kappa}{\log\left(\frac{M^2}{L^2\epsilon}\right)}, \sqrt[6]{\frac{n}{\overline{\Delta}_C}}\right\}\right).$$

Then, KroMagnon*, with step-size* $\gamma = \mathcal{O}(1)\frac{1}{L\kappa}$ *and epoch size* $S = \mathcal{O}(1)\kappa^2$*, attains* $\mathbb{E}\|\mathbf{y}_E - \mathbf{x}^*\|^2 \leq \epsilon$ *after*

$$E = \mathcal{O}(1)\log\left(\frac{a_0}{\epsilon}\right)$$

epochs, where $\mathbf{y}_E$ *is the last iterates of the final epoch, and* $a_0 = \|\mathbf{x}_0 - \mathbf{x}^*\|^2$ *is the initial distance squared to the optimum.*

We would like to note, that the total number of iterations in the above bound is—up to a universal constant—the same as that of serial sparse SVRG as presented in Theorem 12. Again, as with HOGWILD! and ASCD, this implies a linear speedup.

Similar to our ASCD analysis, we would like to remark that between the two bounds on τ, the second one is the more restrictive. The first one, is up to logarithmic factors, equal to the square root of the total number of iterations per epoch; we expect that the size of the epoch is proportional to n, the number of function terms (or data points). This implies that the first bound is for all reasonable applications proportional to $\tilde{\mathcal{O}}(\sqrt{n})$. The second bound which is the more restrictive one is certainly loose; we argue that it can be tightened using a more refined analysis as is explained in our Appendix.

5.3 Proof of Theorem 13

It is easy to see that due to Lemma 9 we get the following bound on the norm of the gradient estimate.

Lemma 14. *For any* k *and* j *we have*

$$\mathbb{E}\left\|\mathbf{v}(\hat{\mathbf{x}}_k, s_k)\right\|^2 \leq 4L^2\left(a_j + a_0 + \mathbb{E}\|\mathbf{x}_j - \hat{\mathbf{x}}_k\|^2\right). \tag{5.5}$$

Proof. Due to Lemma 9 we have $\mathbb{E}\|\mathbf{v}(\hat{\mathbf{x}}_j, s_j)\|^2 \leq 2L^2\mathbb{E}\|\hat{\mathbf{x}}_j - \mathbf{x}^*\|^2 + 2L^2\mathbb{E}\|\mathbf{y} - \mathbf{x}^*\|^2$. Then, using the fact that $\mathbf{y} = \mathbf{x}_0$ and applying the triangle inequality yields the result. □

The set $\mathcal{S}_r^j$ is defined as in the previous section: $\mathcal{S}_r^j = \{\max\{j-r\tau, 0\}, \ldots, j-1, j, j+1, \ldots, \min\{j+r\tau, T\}\}$, and has cardinality at most $2r\tau + 1$. By Assumption 3, there exist diagonal sign matrices $\mathbf{S}_i^j$ with diagonal entries in $\{-1, 0, 1\}$ such that

$$\hat{\mathbf{x}}_k - \mathbf{x}_j = \gamma \sum_{i \in \mathcal{S}_{\ell+1}^j} \mathbf{S}_i^j \mathbf{v}(\hat{\mathbf{x}}_i, s_i). \tag{5.6}$$

This leads to the following lemma.

Lemma 15. *3 Let* $G_r = \max_{k \in \mathcal{S}_r^j} \mathbb{E}\left\|\mathbf{v}(\hat{\mathbf{x}}_k, s_k)\right\|^2$ *and* $\Delta_r = \max_{k \in \mathcal{S}_r^j} \mathbb{E}\|\hat{\mathbf{x}}_k - \mathbf{x}_j\|^2$. *Then,*

$$G_r \leq 4L^2\left(a_j + a_0 + \Delta_r\right) \text{ and } \Delta_r \leq (3\gamma\tau(r+1))^2 G_{r+1}. \tag{5.7}$$

Proof. The proof for the bound on Δ_r is identical to the proof of Lemma 6. We then use Lemma 14 to bound $\mathbb{E}\left\|\mathbf{v}(\hat{\mathbf{x}}_k, s_k)\right\|^2$. □

As explained in the remark after Lemma 6, it should be possible to improve τ^2 to τ in the upper bound on Δ_r. Doing so would improve the condition $\tau = \mathcal{O}\left(\sqrt[6]{n/\overline{\Delta}_{\mathrm{C}}}\right)$ of Theorem 13 to $\tau = \mathcal{O}\left(\sqrt[4]{n/\overline{\Delta}_{\mathrm{C}}}\right)$. The ideas of how one might approach this can be found in Subsection A.2.2 of the Appendix.

We can now obtain bounds on the errors due to asynchrony. The proofs for the following two lemmas can be found in Appendix A.2

Lemma 16. *Suppose* $\tau \leq \frac{\kappa}{\ell}$ *and* $\gamma = \frac{\theta}{12L\kappa}$. *Then* KROMAGNON*'s error terms* R_0^j *and* R_1^j *satisfy the following inequalities.*

$$R_0^j \leq \mathcal{O}(1)\left(L^2(a_j + a_0) + \theta^{2\ell}M^2\right) \quad \text{and} \quad R_1^j \leq \mathcal{O}(1)\left(\theta^2(a_j + a_0) + \theta^{2\ell}\frac{M^2}{L^2}\right). \tag{5.8}$$

Similarly to the ASCD derivations, we obtain the following bound for R_2^j.

Lemma 17. *Suppose* $\tau \leq \frac{\kappa}{\ell}$ *and* $\tau = \mathcal{O}\left(\sqrt[6]{\frac{n}{\overline{\Delta}_C}}\right)$, *and let* $\gamma = \frac{\theta}{12L\kappa}$. *Then* KROMAGNON*'s error term* R_2^j *satisfies the following inequality.*

$$R_2^j \leq \mathcal{O}(1)\left(\theta \cdot m \cdot (a_j + a_0) + \theta^{2\ell}\frac{M^2}{L\kappa}\right).$$

Problem	Dataset	# data	# features	sparsity	Comments
Linear regression	Synthetic	3M	10K	20	
Logistic regression	Synthetic	3M	10K	20	
	rcv1 [33] from [34]	$\approx$ 700K	$\approx$ 42K	$\approx$ 73	Reuters news articles in 2 classes
	url [35] from [34]	$\approx$ 3.2M	$\approx$ 2.4M	$\approx$ 116	Web URL data for suspicious websites
Vertex cover[7]	eswiki-2013 [36–38]	$\approx$ 970K	$\approx$ 23M	$\approx$ 24	Link graph of English part of Wikipedia
	wordassociation-2011 [36–38]	$\approx$ 10.6K	$\approx$ 72K	$\approx$ 7	Free word association graph

Table 1: Problems and datasets used for testing KroMagnon.

5.3.1 Putting it all together

After plugging in the upper bounds on R_0^j, R_1^j, and R_2^j in the main recursion satisfied by KroMagnon, we see that its iterate differences satisfy the inequality:

$$\begin{aligned} a_{j+1} \leq & \left(1 - \gamma m + \mathcal{O}(1)\left(\gamma^2 L^2 + \gamma m \theta^2 + \gamma \theta m\right)\right) a_j + \\ & + \mathcal{O}(1)\left(\gamma^2 L^2 + \gamma m \theta^2 + \gamma \theta m\right) a_0 + \gamma^2 \mathcal{O}(1) \theta^{2\ell} M^2 + \gamma \mathcal{O}(1) \theta^{2\ell} \frac{M^2}{L\kappa} \end{aligned}$$

If we set $\gamma = \mathcal{O}(1)\frac{\theta}{L\kappa}$, *i.e.*, order-wise the same stepsize as serial sparse SVRG (Theorem 12), then the above becomes

$$\begin{aligned} a_{j+1} &\leq \left(1 - \mathcal{O}(1)\frac{\theta}{\kappa^2}\right) a_j + \mathcal{O}(1)\frac{\theta^2}{\kappa^2} a_0 + \theta^{2\ell+1}\mathcal{O}(1)\frac{M^2}{L^2\kappa^2} \\ &\leq \left[\left(1 - \mathcal{O}(1)\frac{\theta}{\kappa^2}\right)^{j+1} + \mathcal{O}(1)\theta\right] a_0 + \theta^{2\ell}\mathcal{O}(1)\frac{M^2}{L^2} \end{aligned}$$

We choose $\theta = \mathcal{O}(1) \leq 1/2$ to be a sufficiently small constant, so that the term $\mathcal{O}(1)\theta$ in the brackets above is at most 0.5. Then we can choose $j = \mathcal{O}(1)\kappa^2$ so that the entire coefficient in the brackets is at most 0.75. Finally, we set

$$\ell = \mathcal{O}(1)\log\left(\frac{M^2}{L^2\epsilon}\right)$$

so that the last term is smaller than $\epsilon/8$. Let $\mathbf{y}_k$ be the iterate after the k-th epoch and $A_k = \mathbb{E}\|\mathbf{y}_k - \mathbf{x}^*\|^2$. Therefore, KroMagnon satisfies the recursion

$$A_{k+1} \leq 0.75 \cdot A_k + \frac{\epsilon}{8} \leq (0.75)^{k+1} A_0 + \frac{\epsilon}{2}.$$

This implies that $\mathcal{O}(1)\log(a_0/\epsilon)$ epochs are sufficient to reach ϵ accuracy, where a_0 is $\|\mathbf{x}_0 - \mathbf{x}^*\|^2$ the initial distance squared to the optimum.

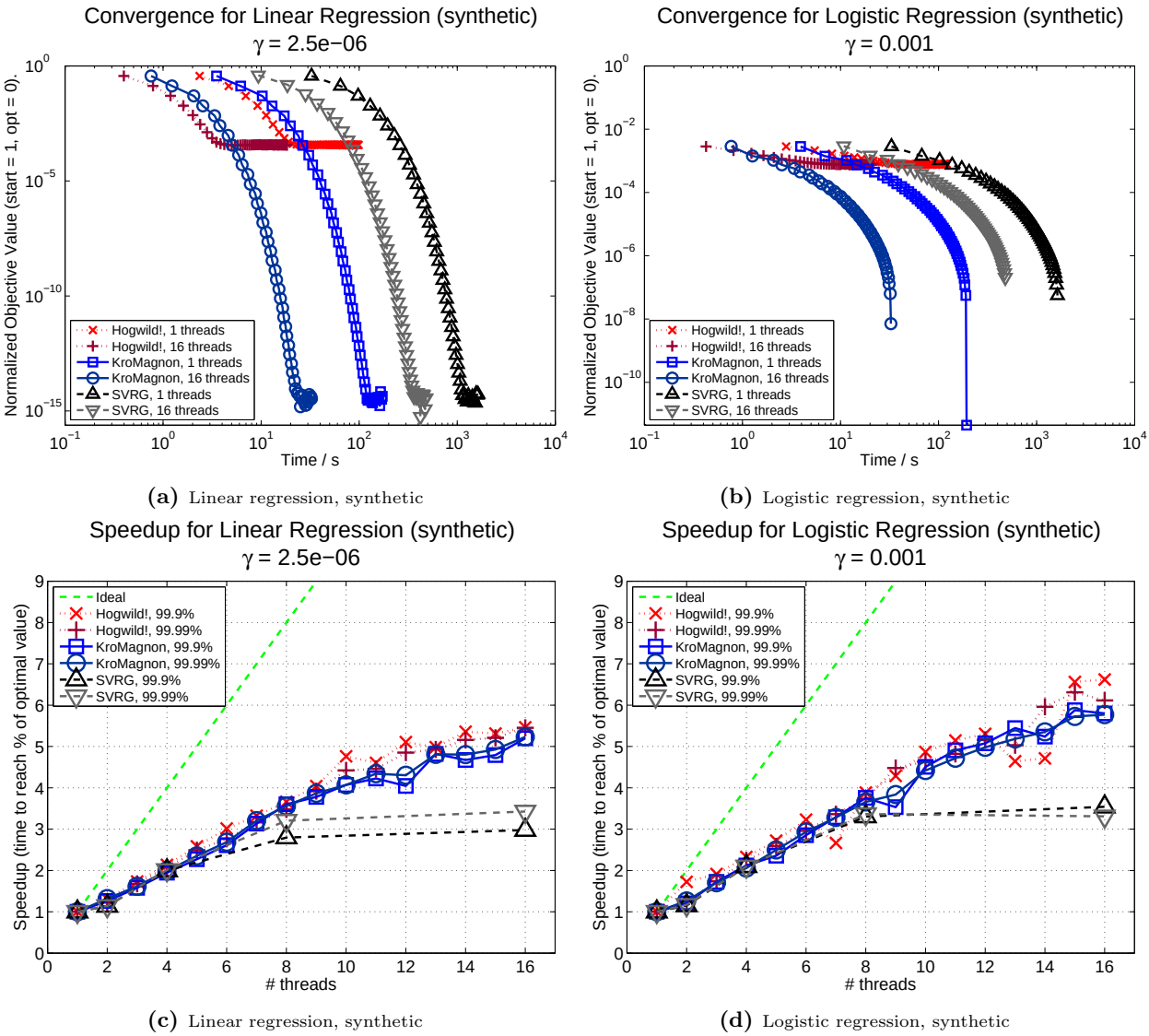
6 Empirical Evaluation of KroMagnon

In this section, we empirically evaluate KroMagnon. Our two goals are to demonstrate that (1) KroMagnon is faster than dense SVRG, and (2) KroMagnon has speedups comparable to those of Hogwild!. We implemented Hogwild!, dense and sparse SVRG in Scala, and tested them on the problems and datasets listed in Table 1. Each algorithm was run for 50 epochs, up to 16 threads. For the SVRG algorithms, we recompute $\mathbf{y}$ and the full gradient $\nabla f(\mathbf{y})$ every 2 epochs. We normalize the objective values such that the objective at the initial starting point has a value of 1, and the minimum attained across all algorithms and epochs has a value of 0. Experiments were conducted on a Linux machine with 2 Intel Xeon Processor E5-2670 (2.60GHz, 8 cores each) with 250Gb memory.

[7] Following [5], we optimize a quadratic penalty relaxation for vertex cover $\min_{x\in[0,1]^{|V|+|E|}} \sum_{v\in V} x_v + \frac{\beta}{2}\sum_{(u,v)\in E}(x_u + x_v - x_{u,v} - 1)^2 + \frac{1}{2\beta}\sum_{v\in V} x_v^2 + \sum_{e\in E} x_e^2$.

Comparison with dense SVRG We were unable to run dense SVRG on the url and eswiki-2013 datasets due to the large number of features. Figures 3a, 3b, 5a show that KroMagnon is 1-2 orders of magnitude faster than dense SVRG. In fact, running dense SVRG on 16 threads is slower than KroMagnon on a single thread. Moreover, as seen in Fig. 4a, KroMagnon on 16 threads can be up to four orders of magnitude faster than serial dense SVRG. Both dense SVRG and KroMagnon attain similar optima.

Speedups We measured the time each algorithm takes to achieve 99.9% and 99.99% of the minimum achieved by that algorithm. Speedups are computed relative to the runtime of the algorithm on a single thread. Although the speedup of KroMagnon varies across datasets, we find that KroMagnon has comparable speedups with Hogwild! on all datasets, as shown in Figure 3c, 3d, 4c, 4d, 5c, 5d.



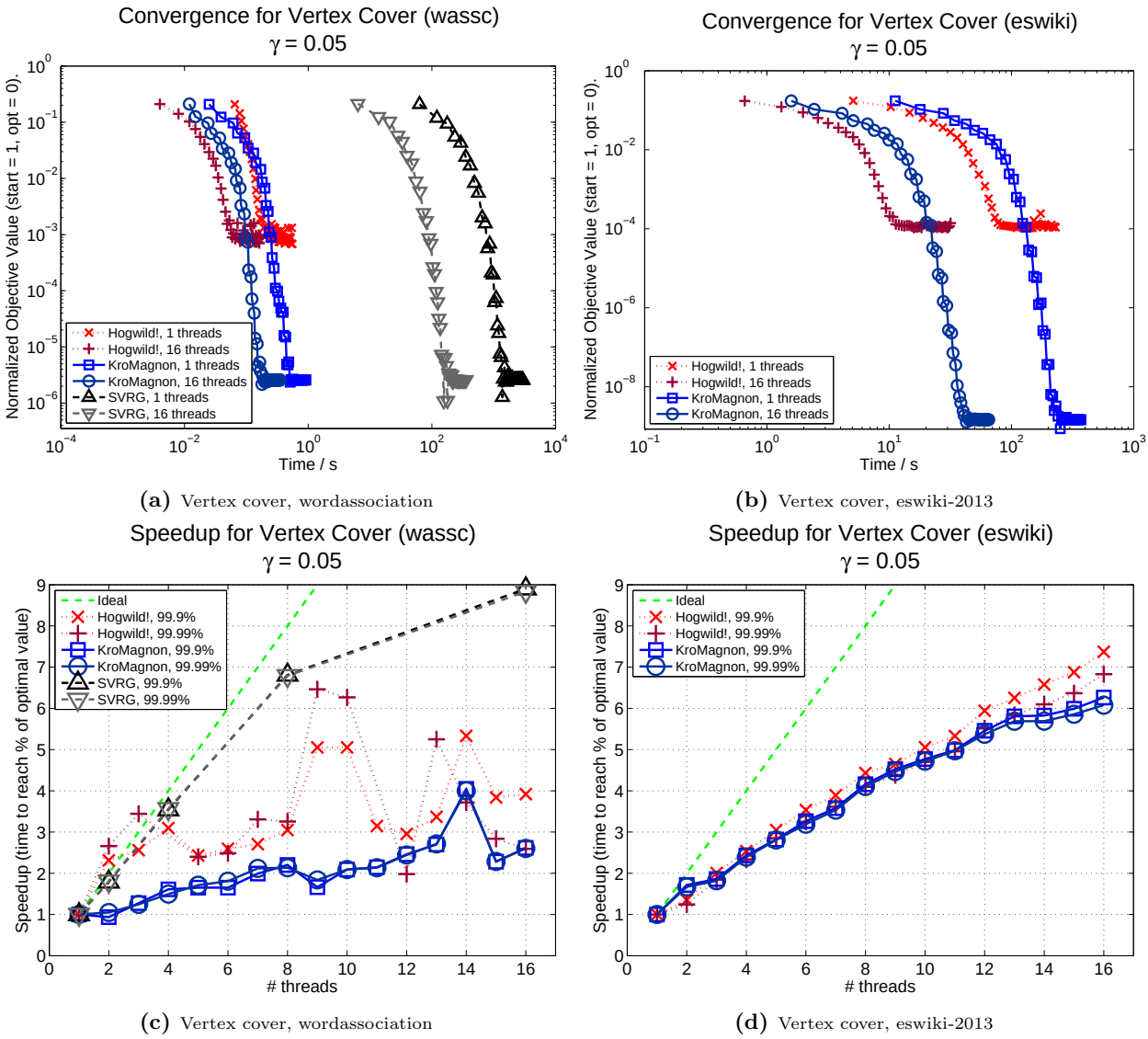
(a) Linear regression, synthetic

(b) Logistic regression, synthetic

(c) Linear regression, synthetic

(d) Logistic regression, synthetic

Figure 3: Convergence and speedup of algorithms on synthetic data.



(a) Vertex cover, wordassociation

(b) Vertex cover, eswiki-2013

(c) Vertex cover, wordassociation

(d) Vertex cover, eswiki-2013

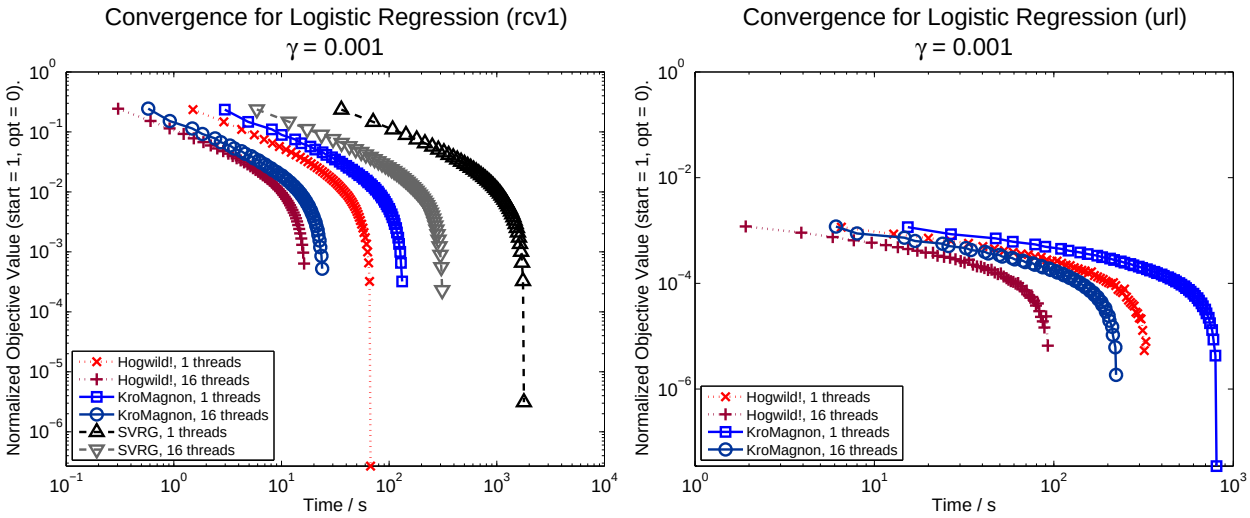
Figure 4: Convergence and speedup of algorithms on the wordassociation, and eswiki-2013 datasets.

7 Conclusions and Open Problems

We have introduced a novel perturbed iterate framework for analyzing parallel asynchronous stochastic optimization algorithms. The main advantage of our framework is that it is straightforward to apply on a range of first order stochastic algorithms, while it involves elementary derivations. Moreover, in our analysis we lift, or relax, some of the assumptions made in prior art, *e.g.*, we do not assume consistent reads, and analyze full gradient updates. We use our framework to analyze HOGWILD! and ASCD, and further introduce and analyze KROMAGNON, a new parallel sparse SVRG algorithm.

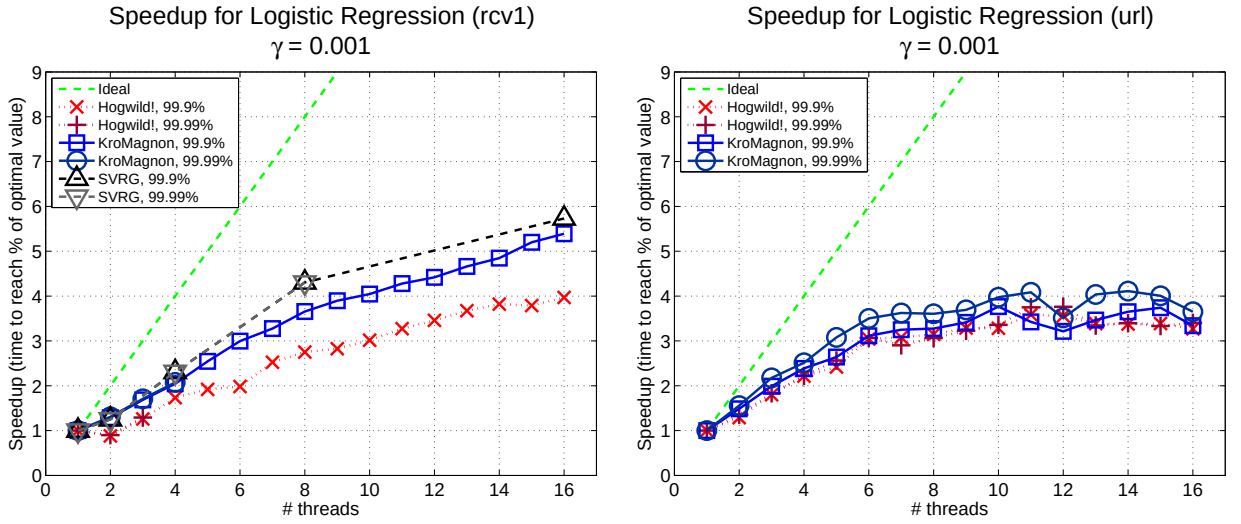
We conclude with some open problems listed below:

1. It would be interesting to obtain tighter bounds for the convergence of function values of the algorithms presented. How do the "errors" due to asynchrony influence the convergence rate of function values? In this case the number of iterations required to reach a target accuracy should scale with the condition number of the objective, not its square. Furthermore, the literature on stochastic coordinate descent establishes convergence results in terms of coordinate wise Lipschitz constants—a more refined smoothness quantity than the full-function smoothness. It would be worthwhile to understand if we



(a) Logistic regression, rcv1

(b) Logistic regression, url



(c) Logistic regression, rcv1

(d) Logistic regression, url

Figure 5: Convergence and speedup of algorithms on the rcv1, and url data sets.

can adapt our framework to take into account these parameters.

2. Our perturbed iterates framework relies fundamentally on the strong convexity assumption. However, asynchronous algorithms are known to perform well on non-strongly convex (and even nonconvex) objectives. Can we generalize our framework to simply convex, or smooth functions?

3. As previously explained, we believe that the upper bounds on τ—the proxy for the number of cores—in our ASCD and KroMagnon analyses seem amenable to significant improvements. It is an open problem to explore the extent of such improvements.

4. Our analysis offers sensible upper bounds only in the presence of sparsity. It seems, however, that to obtain speedup results for Hogwild!, it is only necessary to have small correlation between randomly sampled gradients. In what practical setups do randomly selected gradients have sufficiently small correlation? Does that immediately imply linear speedups in the same way that update sparsity does?

5. In this work we analyzed a variety of very similar stochastic first-order methods. It is an open problem to apply our framework and provide an elementary analysis for a greater variety of stochastic optimiza-

tion algorithms, such as AdaGrad-type schemes (similar to [6]), or stochastic dual coordinate methods (similar to [8]).

6. Capturing the effects of asynchrony as noise on the algorithmic input seems to be applicable to settings beyond stochastic optimization. As shown recently for a combinatorial graph problem, a similar viewpoint enables the analysis of an asynchronous graph clustering algorithm [39]. It is an interesting endeavor to explore the extent to which a perturbed iterate viewpoint is suitable for analyzing general asynchronous iterative algorithms.

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A Omitted Proofs

A.1 ASCD

A.1.1 Bounding R_0^j and R_1^j

Lemma 7. *Let $\tau \leq \frac{\kappa\sqrt{d}}{\ell}$ and set $\gamma = \frac{\theta}{6dL\kappa}$ for any fixed $\theta \leq 1$ and $\ell \geq 1$. Then,*

$$R_0^j \leq \mathcal{O}(1)\left(dL^2 a_j + \theta^{2\ell} dM^2\right) \text{ and } R_1^j \leq \mathcal{O}(1)\left(\theta^2 a_j + \theta^{2\ell}\frac{M^2}{L^2}\right).$$

Proof. Let $A = 2dL^2 a_j$, $B = 2dL^2$ and $C = (\gamma\tau)^2$. Then, we can rewrite the bounds of Lemma 6 as

$$G_r \leq A + B \cdot \Delta_r \text{ and } \Delta_r \leq 3^2(r+1)^2 \cdot C \cdot G_{r+1}, \tag{A.1}$$

which implies $G_r \leq A + 3^2(r+1)^2 BC \cdot G_{r+1}$. We can now upper bound $R_0^j = G_0$, by applying the previous inequality ℓ times. If we expand the formulas, we get

$$R_0^j = G_0 \leq A\sum_{i=0}^{\ell-1}(3^i \cdot i!)^2 (BC)^i + (3^\ell \cdot \ell!)^2 (BC)^\ell G_\ell. \tag{A.2}$$

Since $\gamma = \frac{\theta}{6dL\kappa}$ and $\tau \leq \frac{\kappa\sqrt{d}}{\ell}$ (the choice of τ is made so that the sum in (A.2) is significantly small), we have

$$BC = 2dL^2\gamma^2\tau^2 \leq 2dL^2 \frac{\theta^2}{6^2 d^2 L^2 \kappa^2}\frac{\kappa^2 d}{\ell^2} \leq \frac{1}{2\cdot 3^2 \cdot \ell^2}.$$

Using the upper bound $k! \leq k^k$ on each term of the sum (A.2), and plugging in the upper bound on BC, we get

$$\sum_{i=0}^{\ell-1}(3^i \cdot i!)^2 (BC)^i \leq \sum_{i=0}^{\ell-1}\frac{(i!)^2}{2^i \ell^{2i}} \leq \sum_{i=0}^{\ell-1}\frac{1}{2^i}\left(\frac{i}{\ell}\right)^{2i} \leq 2.$$

Similarly, we obtain the following upper bound on the last term of Eq. A.2

$$\left(3^\ell \cdot \ell!\right)^2 (BC)^\ell \leq 2^{-\ell}\theta^{2\ell}.$$

Finally, $G_\ell \leq dM^2$, and combining the above gives us $R_0^j \leq \mathcal{O}(1)\left(dL^2 a_j + \theta^{2\ell} dM^2\right)$.

We can now bound R_1^j. By definition $R_1^j = \Delta_0$, and from Lemma 6 we have $\Delta_0 \leq 3^2 \cdot C \cdot G_1$. We can bound G_1 similarly to G_0 as

$$G_1 \leq A\sum_{i=0}^{\ell-1}(3^i \cdot (i+1)!)^2 (BC)^i + (3^\ell \cdot (\ell+1)!)^2 (BC)^\ell G_{\ell+1}.$$

As before $BC \leq \frac{\theta^2}{2\cdot 3^2\ell^2}$. Since $(i+1)! \leq 2\ell^i$ for any $0 \leq i \leq \ell$, it follows that

$$\sum_{i=0}^{\ell-1}(3^i \cdot (i+1)!)^2 (BC)^i \leq \mathcal{O}(1),$$

and $(3^\ell \cdot (\ell+1)!)^2 (BC)^\ell \leq \mathcal{O}(1)\theta^{2\ell}$. Therefore, because $G_{\ell+1} \leq dM^2$, we obtain $G_1 \leq \mathcal{O}(1)(dL^2 a_j + \theta^{2\ell} dM^2)$. Since $C = (\gamma\tau)^2 \leq \frac{\theta^2}{dL^2}$, it follows that $R_1 \leq \mathcal{O}(1)\left(\theta^2 a_j + \theta^{2\ell}(M/L)^2\right)$. □

A.1.2 Bounding R_2^j

Lemma 8. *Assume $\tau \leq \frac{\kappa\sqrt{d}}{\ell}$ and $\tau = \mathcal{O}(\sqrt[6]{d})$. Then,*

$$R_2^j \leq \mathcal{O}(1)\left(\theta \cdot m \cdot a_j + \theta^{2\ell}\frac{M^2}{L\kappa}\right).$$

Proof. From (4.5) we can upper bound R_2^j as follows.

$$R_2^j = \mathbb{E}\left\langle \hat{\mathbf{x}}_j - \mathbf{x}_j, \mathbf{g}(\hat{\mathbf{x}}_j, s_j)\right\rangle \leq \gamma \cdot \sum_{\substack{i=j-\tau \\ i\neq j}}^{j+\tau} \mathbb{E}\|\mathbf{g}(\hat{\mathbf{x}}_i, s_i)\| \cdot \|\mathbf{g}(\hat{\mathbf{x}}_j, s_j)\| \cdot \mathbb{1}(s_i = s_j). \tag{A.3}$$

The random variable $\mathbb{1}(s_i = s_j)$ encodes the sparsity of the gradient steps. To take advantage of this sparsity we use smoothness to replace the iterates $\hat{\mathbf{x}}_i$ and $\hat{\mathbf{x}}_j$, by $\hat{\mathbf{x}}_{j-3\tau}$. The latter iterate is independent of both s_i and s_j by our assumption that no more than τ coordinates can be updated while a core is processing a single coordinate. This independence will allows us to "detangle" the expectation of $\mathbb{1}(s_i = s_j)$ from the inner products in the above sum, which will result in a significantly improved bound on R_2^j compared to applying Cauchy-Schwartz directly on it.

For clarity, we note that when $j < 3\tau$, we have $\hat{\mathbf{x}}_{j-3\tau} = \mathbf{x}_0$. From the L-Lipschitz assumption on the gradient $\nabla f(\mathbf{x})$, we get the following bounds

$$\|\mathbf{g}(\hat{\mathbf{x}}_j, s_j)\| \leq \|\mathbf{g}(\hat{\mathbf{x}}_{j-3\tau}, s_j)\| + dL\|\hat{\mathbf{x}}_{j-3\tau} - \hat{\mathbf{x}}_j\| \quad \text{and} \quad \|\mathbf{g}(\hat{\mathbf{x}}_i, s_i)\| \leq \|\mathbf{g}(\hat{\mathbf{x}}_{j-3\tau}, s_i)\| + dL\|\hat{\mathbf{x}}_{j-3\tau} - \hat{\mathbf{x}}_i\|.$$

Then, the expectation of a term $\|\mathbf{g}(\hat{\mathbf{x}}_i, s_i)\| \cdot \|\mathbf{g}(\hat{\mathbf{x}}_j, s_j)\| \cdot \mathbb{1}(s_i = s_j)$ in the sum (A.3) is upper bounded by

$$\mathbb{E}\Big\{\Big(\|\mathbf{g}(\hat{\mathbf{x}}_{j-3\tau}, s_i)\|\,\|\mathbf{g}(\hat{\mathbf{x}}_{j-3\tau}, s_j)\| + (dL)^2\|\hat{\mathbf{x}}_{j-3\tau} - \hat{\mathbf{x}}_i\|\,\|\hat{\mathbf{x}}_{j-3\tau} - \hat{\mathbf{x}}_j\| \\ + dL\|\mathbf{g}(\hat{\mathbf{x}}_{j-3\tau}, s_j)\|\,\|\hat{\mathbf{x}}_{j-3\tau} - \hat{\mathbf{x}}_i\| + dL\|\mathbf{g}(\hat{\mathbf{x}}_{j-3\tau}, s_i)\|\,\|\hat{\mathbf{x}}_{j-3\tau} - \hat{\mathbf{x}}_j\|\Big) \cdot \mathbb{1}(s_i = s_j)\Big\}.$$

We first bound the second term using Cauchy-Schwartz and the property of iterated expectation, to exploit the expectation of the $\mathbb{1}(s_i = s_j)$ term

$$\begin{aligned}
\mathbb{E}\{\|\hat{\mathbf{x}}_{j-3\tau} - \hat{\mathbf{x}}_i\| \cdot \|\hat{\mathbf{x}}_{j-3\tau} - \hat{\mathbf{x}}_j\| \cdot \mathbb{1}(s_i = s_j)\} &\leq \sqrt{\mathbb{E}\{\|\hat{\mathbf{x}}_{j-3\tau} - \hat{\mathbf{x}}_i\|^2\} \cdot \mathbb{E}\{\|\hat{\mathbf{x}}_{j-3\tau} - \hat{\mathbf{x}}_j\|^2 \cdot \mathbb{1}(s_i = s_j)\}} \\
&= \sqrt{\mathbb{E}\{\|\hat{\mathbf{x}}_{j-3\tau} - \hat{\mathbf{x}}_i\|^2\} \cdot \mathbb{E}_{\sim s_j}\{\|\hat{\mathbf{x}}_{j-3\tau} - \hat{\mathbf{x}}_j\|^2 \cdot \mathbb{E}_{s_j}\{\mathbb{1}(s_i = s_j)\}\}} \\
&= \sqrt{\frac{1}{d}}\sqrt{\mathbb{E}\{\|\hat{\mathbf{x}}_{j-3\tau} - \hat{\mathbf{x}}_i\|^2\} \cdot \mathbb{E}\{\|\hat{\mathbf{x}}_{j-3\tau} - \hat{\mathbf{x}}_j\|^2\}} \\
&\leq \mathcal{O}(1)\sqrt{\frac{1}{d}} \cdot \gamma^2\tau^2 \underbrace{\max_{k\in\mathcal{S}_4^j} \mathbb{E}\{\|\mathbf{g}(\hat{\mathbf{x}}_k, s_k)\|^2\}}_{G_4}
\end{aligned}$$

the first equality follows due to $\hat{\mathbf{x}}_j$ being independent of s_j; hence the expectation with respect to s_j can be applied to the indicator function. The last inequality follows from our arguments in the proof of Lemma 6

because both mismatches $\hat{\mathbf{x}}_{j-3\tau} - \hat{\mathbf{x}}_i$ and $\hat{\mathbf{x}}_{j-3\tau} - \hat{\mathbf{x}}_j$ can be written as linear combinations of gradient steps indexed by $\mathcal{S}_4^j$ as in (4.5). Similarly the third term satisfies the inequality

$$\begin{aligned}\mathbb{E}\{\|\mathbf{g}(\hat{\mathbf{x}}_{j-3\tau}, s_j)\| \cdot \|\hat{\mathbf{x}}_{j-3\tau} - \hat{\mathbf{x}}_i\| \cdot \mathbb{1}(s_i = s_j)\} &\leq \sqrt{\mathbb{E}\{\|\mathbf{g}(\hat{\mathbf{x}}_{j-3\tau}, s_j)\|^2\} \cdot \mathbb{E}\{\|\hat{\mathbf{x}}_{j-3\tau} - \hat{\mathbf{x}}_i\|^2 \cdot \mathbb{1}(s_i = s_j)\}} \\ &= \mathcal{O}(1)\sqrt{\frac{1}{d}} \cdot \gamma\tau G_4.\end{aligned}$$

The same bound applies for the fourth term $\mathbb{E}\{\|\mathbf{g}(\hat{\mathbf{x}}_{j-3\tau}, s_i)\| \cdot \|\hat{\mathbf{x}}_{j-3\tau} - \hat{\mathbf{x}}_j\| \cdot \mathbb{1}(s_i = s_j)\}$, while the first term can be easily bounded as

$$\mathbb{E}\{\|\mathbf{g}(\hat{\mathbf{x}}_{j-3\tau}, s_j)\| \cdot \|\mathbf{g}(\hat{\mathbf{x}}_{j-3\tau}, s_i)\| \cdot \mathbb{1}(s_i = s_j)\} \leq \sqrt{\frac{1}{d}} G_4.$$

Putting all pieces together, and using the prescribed value of $\gamma = \frac{\theta}{6dL\kappa}$, we have that

$$R_2 \leq \mathcal{O}(1)\sqrt{\frac{1}{d}}(\gamma\tau)\left(1 + dL\gamma\tau + (dL\gamma\tau)^2\right) G_4 \leq \mathcal{O}(1)\sqrt{\frac{1}{d}} \cdot \gamma \cdot \tau^3 \cdot G_4.$$

The first inequality follows because we are summing over 2τ terms in (A.3). To see why the second inequality is true, note that $dL\gamma \leq \frac{\theta}{6\kappa} \leq 1$ (it is always true that the condition number $\kappa \geq 1$). Therefore

$$1 + dL\gamma\tau + (dL\gamma\tau)^2 \leq 1 + \tau + \tau^2 \leq 3\tau^2.$$

As in the proof of Lemma 7, we can bound G_4 by

$$G_4 \leq \mathcal{O}(1)A\sum_{i=0}^{\ell-1}(3^i \cdot (i+4)!)^2(BC)^i + \mathcal{O}(1)(3^\ell \cdot (\ell+4)!)^2(BC)^\ell G_{\ell+4} \leq \mathcal{O}(1)(dL^2 a_j + \theta^{2\ell} dM^2).$$

The result follows assuming $\tau = \mathcal{O}(\sqrt[6]{d})$ and $\gamma = \frac{\theta}{6dL\kappa}$. □

Remark 2. *We believe that if we use the same bounding technique that we applied for R_2^j on R_0^j and R_1^j, then we can significantly improve the restrictive bound on τ.*

A.2 KroMagnon

A.2.1 Bounding R_0^j and R_1^j

Lemma 16. *Suppose $\tau \leq \frac{\kappa}{\ell}$ and $\gamma = \frac{\theta}{12L\kappa}$. Then* KroMagnon*'s error terms R_0^j and R_1^j satisfy the following inequalities.*

$$R_0^j \leq \mathcal{O}(1)\left(L^2(a_j + a_0) + \theta^{2\ell}M^2\right) \quad \text{and} \quad R_1^j \leq \mathcal{O}(1)\left(\theta^2(a_j + a_0) + \theta^{2\ell}\frac{M^2}{L^2}\right). \tag{5.8}$$

Proof. Let $A = 4L^2(a_j + a_0)$, $B = 4L^2$, and $C = (\gamma\tau)^2$. Then, the inequalities derived above can be rewritten as

$$G_r \leq A + B\Delta_r \text{ and } \Delta_r \leq 3^2(r+1)^2 C G_{r+1}. \tag{A.4}$$

If we expand the formulas, we get for R_0^j the following upper bound

$$R_0^j = G_0 \leq A\sum_{i=0}^{\ell-1}(3^i \cdot i!)^2(BC)^i + (3^\ell \cdot \ell!)^2(BC)^\ell G_\ell.$$

We chose $\gamma = \frac{\theta}{12L\kappa}$ and assumed that $\tau \leq \kappa/\ell$, where $\kappa = \frac{L}{m}$ is the condition number and $\theta \leq 1$. We chose γ to be proportional to the step-size of the serial SVRG, and the assumption on τ is made so that the sum in the above inequality is significantly is significantly small. Then,

$$(3^i \cdot i!)^2(BC)^i \leq (3i)^{2i}\left(4L^2\frac{\theta^2}{4^2 \cdot 3^2 L^2\kappa^2}\frac{\kappa^2}{\ell^2}\right)^i \leq \frac{\theta^2}{4^i}\left(\frac{i}{\ell}\right)^{2i}.$$

and hence

$$\sum_{i=0}^{l-1}(3^i \cdot i!)^2 (BC)^i \leq \sum_{i=0}^{\infty} 2^{-2i} \leq 2.$$

As in the previous sections we assume a uniform upper bound $M > 0$ on the size of the gradient steps: $\max_j \mathbb{E}\|\mathbf{v}(\hat{\mathbf{x}}_j, s_j)\|^2 = M^2$. Therefore

$$R_0^j = G_0 \leq \mathcal{O}(1)\left(L^2(a_j + a_0) + \theta^{2\ell} M^2\right).$$

After an analogous derivation one can see that

$$R_1^j = \Delta_0 \leq \mathcal{O}(1)\left(\theta^2(a_j + a_0) + \theta^{2\ell}\frac{M^2}{L^2}\right),$$

and thus we obtain the result. □

A.2.2 Bounding R_2^j

Lemma 17. *Suppose $\tau \leq \frac{\kappa}{\ell}$ and $\tau = \mathcal{O}\left(\sqrt[6]{\frac{n}{\overline{\Delta}_C}}\right)$, and let $\gamma = \frac{\theta}{12L\kappa}$. Then* KroMagnon*'s error term R_2^j satisfies the following inequality.*

$$R_2^j \leq \mathcal{O}(1)\left(\theta \cdot m \cdot (a_j + a_0) + \theta^{2\ell}\frac{M^2}{L\kappa}\right).$$

Proof. From (5.6) we can upper bound R_2^j as follows.

$$R_2^j = \mathbb{E}\left\langle \hat{\mathbf{x}}_j - \mathbf{x}_j, \mathbf{v}(\hat{\mathbf{x}}_j, s_j)\right\rangle \leq \gamma \cdot \sum_{i=j-\tau, i\neq j}^{j+\tau} \mathbb{E}\|\mathbf{v}(\hat{\mathbf{x}}_i, s_i)\| \cdot \|\mathbf{v}(\hat{\mathbf{x}}_j, s_j)\| \cdot \mathbb{1}(s_i \cap s_j \neq \emptyset) \tag{A.5}$$

The random variable $\mathbb{1}(s_i \cap s_j \neq \emptyset)$ encodes the sparsity of the gradient steps. As in the proof of Lemma 8, we use replace $\hat{\mathbf{x}}_i$ and $\hat{\mathbf{x}}_j$ in the above sum by $\hat{\mathbf{x}}_{j-3\tau}$. For clarity, we not that when $j < 3\tau$, we have $\hat{\mathbf{x}}_{j-3\tau} = \mathbf{x}_0$. Since f_{e_i} are L-smooth, we have

$$\|\mathbf{v}(\hat{\mathbf{x}}_j, s_j)\| \leq \|\mathbf{v}(\hat{\mathbf{x}}_{j-3\tau}, s_j)\| + L\|\hat{\mathbf{x}}_{j-3\tau} - \hat{\mathbf{x}}_j\| \quad \text{and} \quad \|\mathbf{v}(\hat{\mathbf{x}}_i, s_i)\| \leq \|\mathbf{v}(\hat{\mathbf{x}}_{j-3\tau}, s_i)\| + L\|\hat{\mathbf{x}}_{j-3\tau} - \hat{\mathbf{x}}_i\|.$$

Then, the expectation of a term $\|\mathbf{v}(\hat{\mathbf{x}}_i, s_i)\| \cdot \|\mathbf{v}(\hat{\mathbf{x}}_j, s_j)\| \cdot \mathbb{1}(s_i \cap s_j)$ in the sum (A.3) is upper bounded by

$$\mathbb{E}\Big\{\Big(\|\mathbf{v}(\hat{\mathbf{x}}_{j-3\tau}, s_i)\|\,\|\mathbf{v}(\hat{\mathbf{x}}_{j-3\tau}, s_j)\| + L^2\|\hat{\mathbf{x}}_{j-3\tau} - \hat{\mathbf{x}}_i\|\,\|\hat{\mathbf{x}}_{j-3\tau} - \hat{\mathbf{x}}_j\| \\ + L\|\mathbf{v}(\hat{\mathbf{x}}_{j-3\tau}, s_j)\|\,\|\hat{\mathbf{x}}_{j-3\tau} - \hat{\mathbf{x}}_i\| + L\|\mathbf{v}(\hat{\mathbf{x}}_{j-3\tau}, s_i)\|\,\|\hat{\mathbf{x}}_{j-3\tau} - \hat{\mathbf{x}}_j\|\Big) \cdot \mathbb{1}(s_i \cap s_j)\Big\}.$$

Then, since $\mathbb{E}\mathbb{1}(s_i \cap s_j \neq \emptyset) \leq \frac{2\overline{\Delta}_C}{n}$ (recall that $\overline{\Delta}_C$ is the average conflict degrees), R_2^j can be shown to satisfy the inequality

$$R_2^j \leq \mathcal{O}(1)\sqrt{\frac{\overline{\Delta}_C}{n}}\gamma\tau^3\left(L^2(a_j + a_0) + \theta^{2\ell}M^2\right)$$

as in the proof of Lemma 8. The conclusion follows because $\tau = \mathcal{O}\left(\sqrt[6]{\frac{n}{\overline{\Delta}_C}}\right)$ and $\gamma = \frac{\theta}{12L\kappa} = \frac{m\theta}{12L^2}$. □

Remark 3. *Similar to ASCD, by using the same bounding technique of R_2^j on R_0^j and R_1^j, we should significantly improve the restrictive bound on τ in the convergence result of* KroMagnon.